Abstract Algebra

CHARACTERISTIC OF A FIELD

Each element of a field has a certain order finite or infinite, as a member of the additive group of the field.

Definition 1 :- The least positive integer n for which

$$na = 0, \quad a \in F \text{ (field)}$$

is called characteristic of a field.

Definition 2 :- A field is said to be with characteristic zero or infinity, if the order of the unity as a member of the additive group is infinite. The characteristic is said to be p if the order of unity is finite and equal to p.

Theorem :- The charactoristic of a field is either 0 or a prime integer p.

Proof :- Suppose F is a field and e is the unity of F. The order of e, as a member of the additive group of F, may be finite or infinite. If the order of e is infinite, then

$$ne = 0 \Leftrightarrow n = 0$$

Thus the characteristic of F is zero.

If the order of e is finite p (say). Then the charactoristic of F is p. We now have to show that p must be prime

Let if possible

$p = p_1 p_2 . \quad p_1 \neq 1, \quad p_2 \neq 1$

we have $0 = pe = (p_1 p_2) e = (p_1 e)(p_2 e)$

$\Rightarrow$ either $p_1 e = 0$ or $p_2 e = 0$

This however, it is impossible for p is the least positive integer such that $pe = 0$. Then p is prime.

Note :- Fields with non-zero characteristic are known as modular fields.

EXTENSION OF A FIELD

Field extension

Let F be a field. A field K is said to be an extension of F if F is a subfield of K.

or

A field extension of a field F is an ordered pair (K, ϕ), where K is a field and ϕ is a monomorphism of F into K.

Suppose K is a field and F is a subfield of K. The injection mapping $i : F \rightarrow K$ defined by $i(x) = x$ $\forall\ x \in F$ is a monomorphism. Hence (K, i) is a field extension of F.