Exp Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $0<p\leq 1$.

Sol. Let $a(x) = \frac{1}{x(\log x)^p}$ & $a_n = \frac{1}{n(\log n)^p}$

So that $a(n) = a_n \quad \forall\ n \in N$

Then $x \geq 2$, $a(x)$ is non-negative, monotonically decreasing and integrable function.

Now. $\int_2^t a(x)\,dx = \int_2^t \frac{dx}{x(\log x)^p}$

if $p \neq 1$

$$\int \frac{dx}{x(\log x)^p} \qquad \text{where } y = \log x$$

$$= (\log x)^{-p+1} \frac{1}{-p+1}$$

$$\int_2^t \frac{dx}{x(\log x)^p} = \frac{1}{-p+1}\left[(\log x)^{-p+1} \right]_2^t$$

$$= \frac{1}{1-p}\left[(\log t)^{1-p} - (\log 2)^{1-p} \right]$$

if $p=1$ $\int \frac{dx}{x \log x}$ where $y = \log x$

$$\int \frac{dy}{y} = \log y = \log \log x$$

if $p=1$

$$\int_2^t \frac{dx}{x\log x} = [\log\log x]_2^t$$

$$= [\log\log t - \log\log 2]$$

$$\therefore \lim_{t\to\infty}\int_2^t a(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{dx}{x(\log x)^p}$$

$$= \begin{cases} \infty & \text{if } p<1 \\ \frac{1}{p-1}(\log 2)^{1-p} & \text{if } p>1 \\ \infty & \text{if } p=1 \end{cases}$$

Thus $\int_2^\infty a(x)\,dx = \int_2^\infty \frac{1}{x(\log x)^p}$

$= \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p>1$

and divergent if $0<p\leq 1$. Hence by Cauchy's Integral Test the given series is convergent if $p>1$ and divergent if $0<p\leq 1$.