* Cauchy's homogeneous Linear Equation :-

An Eq^n of the form-:

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots\cdots k_{n-1} x \frac{dy}{dx} + k_n y = X \quad \text{--- (1)}$$

where X is a function of x, is called cauchy's homogen. Linear Eq^n with constant coefficients $(k_1, k_2, \cdots k_n)$.

Such equations can be reduced to linear diff. Eq^n with constant coefficients by putting

$$x = e^t \Rightarrow t = \log x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$$

If $D = \frac{d}{dt}$

Then $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$x \frac{dy}{dx} = \frac{d}{dt} y$$

i.e. $$x \frac{dy}{dx} = Dy \qquad \left(\because \frac{d}{dt} = D\right)$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2} \cdot \frac{dy}{dt} + \frac{1}{x} \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x} \cdot \frac{d^2y}{dt^2}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2 \frac{d^2y}{dx^2} = (D^2 - D)y = D(D-1)y.$$

Similarly, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

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Then, $x^n\frac{d^ny}{dx^n} = [D(D-1)(D-2)\cdots\cdots(D-n)]y$

After making these substitutions in cauchy's homogeneous linear diff. Eqn. (1), then these results a linear diff. Eqn with constant Co-efficients. We can solve as:-

Exp. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Solution Given Eqn is cauchy's homogeneous linear diff. Eqn.

put $x = e^t \Rightarrow t = \log x \quad \because \frac{dt}{dx} = \frac{1}{x}$

Then $x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \because D = \frac{d}{dt}$

Then Eqn be comes

$$D(D-1)y - Dy - y = t$$

$$(D-1)^2y = t$$

Which is linear diff. Eqn with constant coeff.

(i) To find C.F

Its Auxilary Eqn is $(D-1)^2 = 0$

$$D = 1, 1$$

$$C.F = (C_1 + C_2 t)e^t = (C_1 + C_2 \log x)e^{\log x}$$

$$C.F = (C_1 + C_2\log x)x.$$

(ii) To find P.I of t.

$$P.I = \frac{1}{(D-1)^2} t$$

$$= (1-D)^{-2} t$$

$$= (1 + 2D + 3D^2 + \dots) t$$

$$= t + 2\frac{d}{dt}t + 3\frac{d^2}{dt^2}t + \dots$$

$$P.I = t + 2 + 0 = t+2 = \log x + 2$$

Hence, the complete solution of Cauchy's homogeneous linear equation (1) is

$$y = C.F + P.I$$

$$y = (c_1 + c_2 \log x)x + \log x + 2$$

Que 1. Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$

Hint ∵ put $x = e^t \Rightarrow t = \log x \Rightarrow x^5 = e^{5t}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$ & $x\frac{dy}{dx} = Dy$ putting in

Eqn. ∵ $D = \frac{d}{dt}$