Definition - Let $f(x)$ be a real valued function with domain $[1, \infty[$.

The function $f(x)$ is said to be non-negative, if
$$f(x) \geq 0 \quad \forall\ x \geq 1$$

The function $f(x)$ is said to be monotonically decreasing, if
$$x \leq y \Rightarrow f(x) \geq f(y); \quad x, y \in [1, \infty[.$$

for example, $f(x) = \frac{1}{x^2}$ is non-negative and monotonically decreasing $\forall\ x \geq 1$

* **Cauchy's Integral Test** :- If $a(x)$ is a non-negative, monotonically decreasing and integrable function such that $a(n) = a_n \ \forall\ n \in N$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} a(x)\,dx$ is convergent.

Proof :- let $a(x)$ is monotonically decreasing

So -
$$a(n) \geq a(x) \geq a(n+1) \quad \text{when } n \leq x \leq n+1$$

Since $a(x)$ is non-negative and integrable

So. $$\int_n^{n+1} a(n)\,dx \geq \int_n^{n+1} a(x)\,dx \geq \int_n^{n+1} a(n+1)\,dx$$

or $a(n)[n+1-n] \geq \int_n^{n+1} a(x)\,dx \geq a(n+1)[n+1-n]$

$$a(n) \geq \int_n^{n+1} a(x)\,dx \geq a(n+1)$$

or

$$a_n \geq \int_n^{n+1} a(x)\,dx \geq a_{n+1} \quad \text{--- (1)} \quad [\because a(n) = a_n]$$

Putting $n = 1, 2, 3 \cdots (n-1)$ and adding

$$a_1 + a_2 + a_3 + \cdots a_{n-1} \geq \int_1^2 a(x)\,dx + \int_2^3 a(x)\,dx + \int_3^4 a(x)\,dx + \cdots \int_{n-1}^{n} a(x)\,dx \geq a_2 + a_3 + \cdots a_n$$

$$S_n - a_n \geq \int_1^n a(x)\,dx \geq S_n - a_1$$

where $S_n = a_1 + a_2 + \cdots a_n$

$$S_n - a_n \geq \int_1^n a(x)\,dx \geq S_n - a_1$$

or

$$S_n - a_1 \leq \int_1^n a(x)\,dx \leq S_n - a_n \quad \text{--- (2)}$$

<u>The condition</u> <u>is necessary</u>:

Suppose the series $\sum_{n=1}^{\infty} a_n$ is converg[ent]

then there exists a positive number K such that $S_n \leq K \;\forall n$ --- (3)

from ② & ③, we get

$$\int_1^n a(x)\,dx \le S_n \le K \quad \forall n$$

$(\because K \text{ is finite.})$

Hence $\int_1^\infty a(x)\,dx$ is convergent

The condition is sufficient

Suppose $\int_1^\infty a(x)\,dx$ is convergent. Then there exists a positive number k such that

$$\int_1^n a(x)\,dx \le k \quad \forall n \quad \text{—— ④}$$

from ② & ④

$$S_n - a_1 \le \int_1^n a(x)\,dx \le k \quad \forall n$$

or $S_n \le k + a_1 \;\; \forall n$

So that $\langle S_n \rangle$ is bounded above

Hence the series $\sum a_n$ is convergent.

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