Topic :- Boundary Conditions For The Electromagnetic Field Vectors

The electromagnetic field vector $\vec{B}, \vec{E}, \vec{D}$ and $\vec{H}$ of the electromagnetic waves satisfy some conditions at the surface ~~of~~ separating two media when it ~~propagates~~ crosses that surface. These conditions are known as "Boundary Condition". These are as follows

(i) The normal component of magnetic induction is continuous across boundary

(ii) The tangent component of $\vec{E}$ is continuous across the interface

(iii) The normal component of electric displacement is discontinuous across the interface.

(iv) The tangential component of magnetic intensity is continuous across the surface separating two dielectrics.

Above boundary conditions are proved by using Maxwell's four equation of electromagnetic field. To do it consider an interface (surface) separating two media 1 and 2 with an outwardly drawn unit vector $\vec{n}$ normal to the infinitesimal area element ds of the surfaces as shown in fig (1)

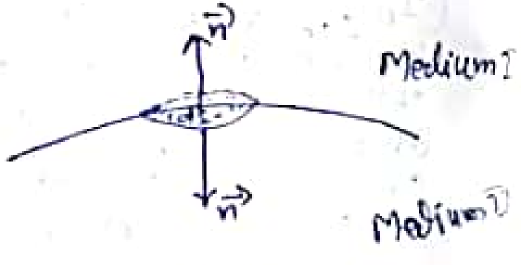

Fig (1)

First Boundary Condition:- Maxwell's equation satisfying the magnetic induction is given by

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \cdots\cdots (1)$$

This equation shows that $\vec{B}$ is solenoidal in characteristic, so we construct a pill box ~~at the surface~~ normal to the interface with surfaces S_1, S_2, S_1' and S_2' as shown in fig (2).

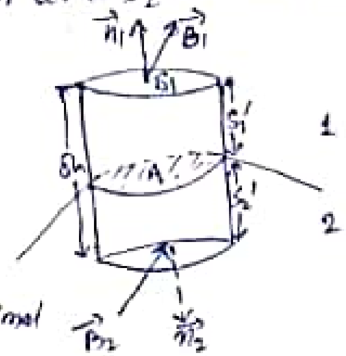

Fig (2)

Using divergence theorem, equation (1) becomes

$$\int \vec{\nabla}\cdot\vec{B}\, dV = 0 \rightarrow (2)$$

where dV is the infinitesimal volume element.

Transforming volume integral into surface integral by Gauss theorem, we get

$$\oint_S \vec{B}\cdot\vec{n}\, ds = 0 \quad \rightarrow (3)$$

Applying equation (3) to the whole surface of pill box, we have

$$\int_{S_1} \vec{B}_1\cdot\vec{n}_1\, ds + \int_{S_2} \vec{B}_2\cdot\vec{n}_2\, ds + \int_{S_1'} \vec{B}_1\cdot\vec{n}_1\, ds + \int_{S_2'} \vec{B}_2\cdot\vec{n}_2\, ds = 0 \quad \rightarrow (4)$$

For only at interface $\delta h \to 0$ and hence the third and fourth term in eq^n(4) will meaningless & $ds \to dA$

$\therefore$ We have

$$\int_{S_1} \vec{B}_1\cdot\vec{n}_1\, dA + \int_{S_2} \vec{B}_2\cdot\vec{n}_2\, dA = 0 \quad \rightarrow (5)$$

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or $\int_A (\vec{B_1}\cdot\vec{n_1} + \vec{B_2}\cdot\vec{n_2})\,dA = 0$

$\therefore$ $\vec{B_1}\cdot\vec{n_1} + \vec{B_2}\cdot\vec{n_2} = 0$ as A is quite arbitrary

$\therefore$ $\vec{B_1}\cdot\vec{n_1} = -\vec{B_2}\cdot\vec{n_2} \longrightarrow (6)$

If $\vec{n}_{12}$ is the unit vector pointing from the first into the second medium, then $\vec{n_1} = -\vec{n}_{12}$ and $\vec{n_2} = \vec{n}_{12}$. Therefore eqn (6) becomes $-\vec{B_1}\cdot\vec{n}_{12} = -\vec{B_2}\cdot\vec{n}_{12}$

$\therefore$ or, $\vec{B_1}\cdot\vec{n}_{12} = \vec{B_2}\cdot\vec{n}_{12}$

or, $\vec{B}_{1n} = \vec{B}_{2n}$

i.e the normal component of magnetic induction is continuous across the boundary.

Second Boundary Condition

To prove this condition, we take Maxwell's equation

$$\text{Curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow (7)$$

Integrating equation (7) over the surface bounded by rectangular loop ABCD, we have

$$\int_S \text{Curl}\,\vec{E}\cdot\vec{n}\,ds = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{n}\,ds \rightarrow (8)$$

According to Stokes theorem, the surface integral of L.H.S can be transformed into a line integral ~~over the path ABCD with the help of~~

$$\int_L E\,dl = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{n}\,ds \longrightarrow (9)$$

To solve the line integral, we construct a rectangular loop ABCD bounding a surface S as shown in fig (2). The loop ABCD has four lines AB, BC, CD, and DA.

For these lines, eqn (9) becomes

$$\int_{ABCD} E\,dl = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{n}\,ds$$

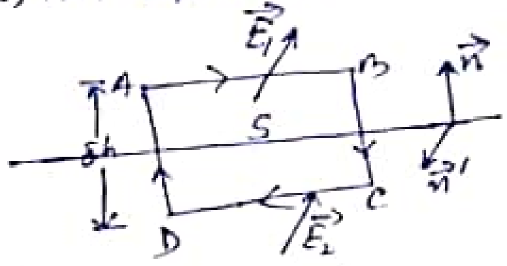

Fig (2)

or $\int_{AB} E_1\,dl + \int_{CD} E_2\,dl$ + Contribution from sides BC and DA

$$= -\int_S \frac{\partial \vec{B}}{\partial t}\,\vec{n}\cdot ds$$

$$= -\frac{\partial \vec{B}}{\partial t}\int \vec{n}\cdot ds$$

$\therefore$ At interface, $\int_{AB} E_1\,dl + \int_{CD} E_2\,dl = 0$ (As $\frac{\partial \vec{B}}{\partial t}$ is constant every where & integration of area vanishes at interface i.e $ds \to 0$)

or, $E_1(AB) + E_2(CD) = 0$

or, $E_1(AB) = -E_2(CD)$

or, $E_1(AB) = E_2(AB)$ $\quad \because \vec{AB} = -\vec{CD}$

or, $E_{1t} = E_{2t}$

Thus the tangential components of $\vec{E}$ are continuous across the interface.

(Continued)

Third Boundary Condition

We take Maxwell's first equation as

$$\operatorname{Div} \vec{D} = \rho$$

Integrating over volume V on both sides, we have

$$\int_V \operatorname{div} \vec{D} = \int_V \rho \, dV \longrightarrow (10)$$

With the help of Gauss divergence theorem transforming volume integral of L.H.S in surface integral, we get

$$\oint_S \vec{D} \cdot \vec{n} \, ds = \int_V \rho \, dv$$

Applying above equation for all surfaces of fig (1), we have

$$\int_{S_1} \vec{D}_1 \cdot \vec{n}_1 \, ds + \int_{S_2} \vec{D}_2 \cdot \vec{n}_2 \, ds + \begin{matrix}\text{Contribution}\\ \text{from walls}\end{matrix} = \int_V \rho \, dv \longrightarrow (11)$$

For the boundary surface, S_1 and S_2 must approach each other so that $S_1 = S_2 = A$ & $\delta h \to 0$

obviously $\int_V \rho \, dv \to \int_A \sigma \, dA = \sigma A$.

Hence equation (11) becomes,

$$\vec{D}_1 \cdot \vec{n}_1 S_1 + \vec{D}_2 \cdot \vec{n}_2 S_2 = \sigma A$$

or, $\vec{D}_1 \cdot \vec{n}_1 + \vec{D}_2 \cdot \vec{n}_2 = \sigma$

or, $\vec{D}_1 \cdot \vec{n}_1 - \vec{D}_2 \cdot \vec{n}_1 = \sigma$

or, $D_{1n} - D_{2n} = \sigma$

Thus the normal component of electric displacement across the interface ~~and~~ separating two media is not continuous.

Fourth Boundary Condition

According to Maxwell's fourth equation, we have

$$\operatorname{Curl} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow (12)$$

Integrating eqn (12) over surface area S, we get

$$\int_S \operatorname{curl} \vec{H} \cdot \vec{n} \, ds = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \vec{n} \, ds \rightarrow (13)$$

Transforming the surface integral into line integral with the help of Stoke's theorem, we have

$$\int_{ABCD} \vec{H} \cdot \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{n} \, ds \longrightarrow (14)$$

Applying eqn (14), in fig (2), only for L.H.S, we have

$$\int_{AB} \vec{H}_1 \cdot \vec{dl} + \int_{CD} \vec{H}_2 \cdot \vec{dl} + \begin{matrix}\text{Contribution from}\\ \text{sides BC and DA}\end{matrix} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{n} \, ds \rightarrow (15)$$

In the limit $h \to 0$, we have

$$\int_{AB} \vec{H}_1 \cdot \vec{dl} + \int_{CD} \vec{H}_2 \cdot \vec{dl} = J_{S\perp} \vec{n}' \longrightarrow (16)$$

where $J_{S\perp}$ is the component of the surface current density perpendicular to the direction of the $\vec{H}$ component which is being matched.

$$H_1 \int_{AB} dl + H_2 \int_{CD} dl = J_{S\perp} n'$$

or $H_1(AB) + H_2(CD) = J_{S\perp} n'$

or $H_{1t}(AB) - H_{2t} AB = J_{S\perp} n'$

or $H_{1t} - H_{2t} = J_{S\perp}$

The surface current density is zero unless the conductivity is infinite, hence for finite conductivity $H_{1t} = H_{2t}$ hence the condition is proved.

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