Cyclotron frequency, Larmor radius, Drift velocity of guiding Centre

To understand the Confinement of Plasma, let us Consider the motion of charged particle in a given magnetic field. For a particle of charge q and mass m, moving with velocity $\vec{v}$ in a region of space where there is only a magnetic induction $\vec{B}$, the equation of motion is

$$m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})$$ which reads in

Cartisian Co-ordinates

$$m\dot{v}_x = qBv_y, \quad m\dot{v}_y = -qBv_x, \quad m\dot{v}_z = 0$$

or $$\ddot{v}_x = \frac{q\vec{B}}{m}\dot{v}_y = -\left(\frac{q\vec{B}}{m}\right)^2 v_x \quad \text{--- (1)}$$

& $$\ddot{v}_y = -\frac{q\vec{B}}{m}\dot{v}_x = -\left(\frac{q\vec{B}}{m}\right)^2 v_y$$

This describes a Simple harmonic oscillator at a frequency $$\boxed{\omega_c = \frac{|q|\vec{B}}{m}} \quad \text{--- (2)}$$ Known as

Cyclotron frequency.

By Convention ω_c is taken always nonnegative. $\vec{B}$ is measured in tesla. The Solution of equation (1) is

$$v_{x,y} = v_\perp \exp(\pm i\omega_c t + i\delta_{x,y}) \quad \text{--- (3)}$$

where $\pm$ denote the sign of charge q. We may choose the phase δ so that

$$v_x = v_\perp e^{i\omega_c t} = \dot{x} \quad \text{--- (4)}$$

where $v_\perp$ is a +ve Constant denoting the speed in the plane $\perp$r to $\vec{B}$. Then

$$v_y = \frac{m}{q\vec{B}} \dot{v}_x = \pm \frac{1}{\omega_c} \dot{v}_x = \pm i v_\perp e^{i\omega_c t} = \dot{y} \quad \text{— (5)}$$

Integrating once again, we have

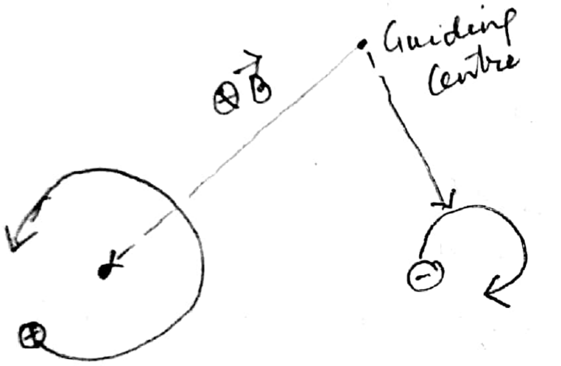
$$x - x_0 = -i \frac{v_\perp}{\omega_c} e^{i\omega_c t}, \quad y - y_0 = \frac{v_\perp}{\omega_c} e^{i\omega_c t} \quad \text{— (6)}$$

we can now define Larmor radius as

$$r_L = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{q\vec{B}} \quad \text{— (7)}$$

Taking the real part of eqn (6), we have

$$x - x_0 = r_L \sin \omega_c t, \quad y - y_0 = \pm r_L \cos \omega_c t \quad \text{— (8)}$$



This describes a circular orbit about a centre called guiding centre (x_0, y_0) which is fixed (fig). The direction of gyration is always such that the magnetic field ~~gen~~ generated by the charged particle is opposite to the externally applied field. In addition to this motion, there is an arbitrary velocity v_z along $\vec{B}$ which is not affected by $\vec{B}$. The trajectory of the ~~(+)~~ charged particle in space is, in general, a helix

If we now apply an electric field $\vec{E}$, in the present situation, the motion will be found to be the sum of two motions: the usual circular Larmor gyration plus a drift of the guiding centre. We now choose $\vec{E}$ to lie in x-y plane so that $E_y = 0$.

As before, the component of velocity is unrelated to the transverse components and can be treated seperately. The equation of motion is now

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \quad \text{—— (9)}$$

whose z- component is

$$\frac{dv_z}{dt} = \frac{q}{m}E_z$$

or

$$v_z = \frac{qE_z}{m}t + v_{z0} \quad \text{—— (10)}$$

This is a straight forward acceleration along $\vec{B}$. The transverse components of eq. (9) are

$$\left.\begin{aligned} \frac{dv_x}{dt} &= \frac{q}{m}E_x \pm \omega_c v_y \\ \frac{dv_y}{dt} &= 0 \mp \omega_c v_x \end{aligned}\right| \quad \text{—— (11)}$$

Differentiating, we have (for const $\vec{E}$)

$$\left.\begin{aligned} \ddot{v}_x &= -\omega_c^2 v_x \\ \ddot{v}_y &= 0 \mp \omega_c\left(\frac{q}{m}\dot{E}_x \pm \omega_c v_y\right) = -\omega_c^2\left(\frac{E_x}{B} + v_y\right) \end{aligned}\right| \quad (12)$$

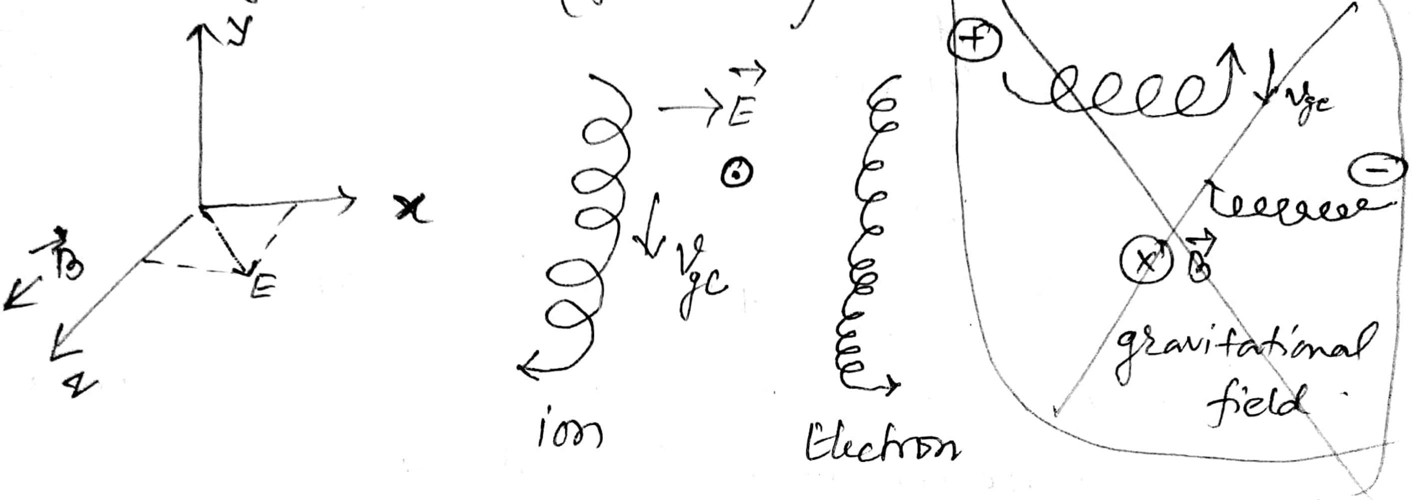
We can write this as

$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2\left(v_y + \frac{E_x}{B}\right)$$

So that eqn (12) is reduced to the previous case if we replace v_y by $v_y + \frac{E_x}{B}$. Equation (5) is therefore replaced by

$$\left.\begin{aligned} v_x &= v_\perp e^{i\omega_c t} \\ v_y &= \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B} \end{aligned}\right| \quad \text{—— (13)}$$

The Larmor motion is the same as before, but there is superimposed a drift velocity of the guiding centre in the $-y$ direction (for $E_x > 0$)



To obtain the general formula for drift velocity $\vec{v}_{gc}$, let us solve eqn. (9) in vector form. We may omit the term $m\frac{d\vec{v}}{dt}$ in eqn. (9), then eqn. (9) becomes

$$\vec{E} + \vec{v} \times \vec{B} = 0 \quad \text{—— (14)}$$

Taking the cross product with $\vec{B}$, we have

$$\vec{E} \times \vec{B} = \vec{B} \times (\vec{v} \times \vec{B}) = \vec{v} B^2 - \vec{B}(\vec{v} \cdot \vec{B}) \quad \text{— (15)}$$

The transverse components of this eqn. are

$$v_{\perp gc} = \vec{E} \times \vec{B} / B^2 = v_E \quad \text{—— (16)}$$

Here v_E is defined as electric field drift of the guiding centre. In magnitude, this drift is

$$v_E = \frac{E}{B} \quad \text{— (17)}$$

It is important to note that v_E is independent of q, m and $v_\perp$.