Application of Hamiltonian to the problem of motion of a charged Particle in an electromagnetic field

The force on a Particle with charge q is given by Lorentz formula

$$F = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right] \quad \text{--- (1)}$$

where the electric and magnetic field in vacuum can be expressed in the form

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad E = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t},$$

where $\vec{A}$ is the vector Potential and ϕ the scalar Potential, c is the velocity of light in vacuum.

The equation of motion is given by

$$\dot{\vec{P}} = \frac{d}{dt}(m\vec{v}) = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

$$= q\left[-\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \frac{\vec{v}}{c}\times(\vec{\nabla}\times\vec{A})\right] \quad \text{--- (2)}$$

we know that $\vec{v}\times(\vec{\nabla}\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$

Thus we have

$$\frac{d}{dt}(m\vec{v}) = q\left[-\nabla\phi - \frac{1}{c}\left(\frac{\partial\vec{A}}{\partial t} + \vec{v}\cdot\nabla\vec{A}\right) + \nabla\frac{\vec{v}\cdot\vec{A}}{c}\right]$$

$$= e\left[-\nabla\left(\phi - \frac{\vec{v}\cdot\vec{A}}{c}\right) - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right]$$

or $$\frac{d}{dt}\left(m\vec{v} + \frac{q}{c}\vec{A}\right) = -q\nabla\left(\phi - \frac{\vec{v}\cdot\vec{A}}{c}\right) \quad \text{--- (3)}$$

This equation has the general form of a set of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

Here
$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial L}{\partial \dot{x}_i} = m\vec{v}_i + \frac{q}{c}A_i \\ \frac{\partial L}{\partial q_k} &= \frac{\partial L}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{-q\left(\phi - \frac{v_k A_k}{c}\right)\right\} \end{aligned}\right\} \quad \text{--- (4)}$$

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Or in vector notation

$$\frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c}\vec{A}$$

$$\& \quad \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left\{-q\left(\phi - \frac{\vec{v}.\vec{A}}{c}\right)\right\} \quad \text{—— (5)}$$

These relation (4) & (5) are true only when

$$L = \frac{1}{2}mv^2 - q\phi + q\frac{\vec{v}.\vec{A}}{c}$$

$$\text{Momentum } \vec{p} = \frac{\partial L}{\partial v} = m\vec{v} + \frac{q}{c}\vec{A} \quad \text{—— (6)}$$

Thus the momentum of a charged Particle moving in electromagnetic field is not defined as its mass times velocity.

From eqn. (6), we may write

$$v = \frac{1}{m}\left[\vec{P} - \frac{q}{c}\vec{A}\right]$$

Hence the Hamiltonian H is given by

$$H = \frac{1}{2m}\left[P - \frac{q}{c}\vec{A}(r,t)\right]^2 + q\phi(r,t) \quad \text{—— (7)}$$

The Canonical equations are then (7), together with

$$\dot{P} = -\vec{\nabla}\vec{H} = -q\vec{\nabla}\phi + \frac{q}{c}\vec{\nabla}(\vec{v}.\vec{A}) \quad \text{—— (8)}$$

This is same as Lorentz force equation

If we write kinetic energy as

$$T = \frac{1}{2}mv^2 = E - q\phi$$

we can write

$$P_k = m\dot{x}_k = P_k - \frac{q}{c}A_k$$

for the kinetic momentum as the difference between the total momentum and e.m. momentum the kinetic momentum is, of course, related to K.E as

$$T = \frac{1}{2m}\sum_k P_k^2 = P^2/2m$$ but is not Canonically Conjugate

i.e. $P_k \neq \frac{\partial H}{\partial x_k}$, $\dot{x}_k \neq \partial H/\partial P_k$

The Lagrangian (nonrelativistic) for a single Particle is

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

with Canonical momenta

$$P_i = mv_i + \frac{q}{c}A_i$$

From the definition H is given by

$$H = \sum_i P_i v_i - L$$

$$= mv^2 + \frac{q}{c}\vec{A}\cdot\vec{v} - L$$

or finally $H = \frac{mv^2}{2} + q\phi = T + q\phi$ the total energy of the Particle.

In terms of momena, the Hamiltonian appears as

$$H = \frac{1}{2m}\left(P - \frac{q}{c}A\right)^2 + q\phi$$

A Similar result is obtained for the relativistic Hamiltonian with electromagnetic forces. There too the Canonical momenta $\left(P_i = mv_i + \frac{q}{c}A_i\right)$ have the additional expression qA_i/c which is just such as to eliminate the term in Hamiltonian involving vector potential. The H is again

$$H = T + q\phi$$

as in the relativistic Case

$$H = \sqrt{\left(\vec{P} - \frac{q\vec{A}}{c}\right)^2 c^2 + m^2c^4} + q\phi$$

The End