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Action-Angle variables + Its Example – the Kepler's Problems

In case, if the time does not appear in Hamiltonian H, then there exists a time-independent transformation generated by $F'(q_k, P_k)$ alone, such that P_k are constants and the new Hamiltonian is not zero but equal to original one i.e. $\bar{H}(Q_K, P_K) = H(q_K, p_K)$, showing that Q_k are not constant. [q_k, p_k are old set of position co-ordinates and momenta & Q_K, P_K are corresponding quantities in new set

H is the Hamiltonian in old set, $\bar{H}$ is the hamiltonian in new set, F the generating function]

Such a transformation generated by the time ~~dependent~~ independent part of S' defined as

$$S' = F'(q_k, P_k) - Et, \quad \text{---} \quad (1)$$

with) $$F' = F + P_k Q_k \quad \ldots \quad (2)$$

[S is Hamilton's characteristic function, E is total energy of the system]

where F' generates a cannonical transformation, accordingly as, $p_k = \dfrac{\partial F'(q_k, P_k)}{\partial q_k}$, $Q_k = \dfrac{\partial F'(q_k, P_k)}{\partial P_k}$,

$$\bar{H}(Q_K, P_K) = H(q_k, p_k) + \frac{\partial F'(q_k, P_k)}{\partial t} \quad \ldots \quad (3)$$

This leads to an idea of action and angle variables.

In order words we sometimes deal with the systems having their motion periodic. Such systems are conveniently dealt with a variation of Hamilton-Jacobi procedure according to which the constants of integration appearing directly in the solution of Hamilton-Jacobi equation are taken as new momenta. These momenta are replaced by J_K which form a set of n independent function α_k's. These J_K' are known as action variables.

In order to develop the idea of action and angle variables, let us consider a particular system with one degree of freedom described by a time independent Hamiltonian $H(q,p)$, such that cannonical transformation given by (3) and generated by $F'(q,P)$ are

$$q = \frac{\partial F'}{\partial p}, \; Q = \frac{\partial F'}{\partial P}, \; \bar{H} = H \qquad (4)$$

Subject to

$$\dot{P} = -\frac{\partial \bar{H}}{\partial Q}, \; \dot{Q} = \frac{\partial \bar{H}}{\partial P} \qquad (5)$$

If F' be so chosen that the new momentum P is constant of motion say $P = J$ and let then $Q = \omega$, where ω is a cyclic co-ordinate, then we have using the idea of Hamilton's characteric function,

$$\bar{H} = \bar{H}(J) = H(q,p) = E \qquad (6)$$

where by using eqn (4), $\dot{Q} = \dot{\omega} = \frac{\partial \bar{H}}{\partial P} = \frac{\partial \bar{H}}{\partial J}$

Since, $\bar{H}$ being the function of J alone, then we have

$$w = \frac{\partial F'}{\partial J} = \frac{\partial F'}{\partial P}, \quad \dot{w} = \frac{d\bar{H}}{dJ} = \nu \text{ (say) and } \dot{j} = -\frac{\partial \bar{H}}{\partial w} \quad \text{— (7)}$$

~~Here $\bar{H}$ and $\bar{J}$~~ Here $\bar{H}$ and J both being constants, ν is also a constant (frequenly) and hence the cyclic coordinate w, which is conjugate to the constant of motion J increases linearly with time. Therefore integration of (7) gives

$$w = \nu t + \epsilon \quad \text{— (8)}$$

where ϵ is a constant of integration

The relation follows that the time-independent part of S' generates a transformation which gives way to a new coordinate ($Q = w$) which increases linearly with time.

If we now consider the case of a no. of systems in which q and p are periodic fns of time then in order to evaluate the action over one periodic, which is constant of the motion, let us chose J in the form of line integral known as phase integral or action variable expressed as

$$J = \oint p\, dq \quad \text{— (9)}$$

Here the integrand p can be expressed in terms of q with the help of (5) and then the integral can be evaluated.

let Δw be the increment in the value of w during one period of motion, then we can express it as

$$\Delta w = \oint dw = \oint \frac{\partial w}{\partial q} dq = \oint \frac{\partial}{\partial q}\left(\frac{\partial F'}{\partial J}\right) dq \quad \text{from eq (7)}$$

$$= \frac{\partial}{\partial J} \oint \frac{\partial F'}{\partial q}\, dq = \frac{\partial}{\partial J} \oint p\, dq \quad \text{from eq (4)}$$

$$= \frac{\partial}{\partial J} J = 1 \text{ from eq. (9)}$$

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Hence the change in ω during a complete Period is unity and from eq (8), taking $\epsilon = 0$, we have $\omega = 1$, if $t = \nu^{-1}$ i.e. ω increases by unity in time ν^{-1} which shows that ν gives the frequency of the Periodic Motion. While dealing with the Problems, J is terms as the action variable and ω is corresp angle variable.

Its Example – Kepler's Problem

Discussed in Next class