Recursion Formula :-

The Hermite's differential equation for function $F(y)$ is given as —

$$\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} + \left(\frac{\alpha}{\beta} - 1\right)F = 0 \quad —(1)$$

This equation may be solved by Polynomial method in which function F can be expanded as power series in y i.e,

$$F = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + \cdots\cdots$$

then $$\frac{dF}{dy} = a_1 + 2a_2 y + 3a_3 y^2 + 4a_4 y^3 + \cdots\cdots$$

& $$\frac{d^2F}{dy^2} = 2a_2 + 6a_3 y + 12a_4 y^2 + \cdots\cdots$$

or $$\frac{d^2F}{dy^2} = (1\times2)a_2 + (2\times3)y a_3 + (3\times4)a_4 y^2 + \cdots\cdots$$

Substituting these equations in eqn — (1), we get

$$\left[(1\times2)a_2 + (2\times3)a_3 y + (3\times4)a_4 y^2 + \cdots\right] - \left[(2\times1)a_1 y + (2\times2)a_2 y^2 + (2\times3)a_3 y^3 + \cdots\right] + \left[\left(\frac{\alpha}{\beta} - 1\right)a_0 + \left(\frac{\alpha}{\beta} - 1\right)a_1 y + \left(\frac{\alpha}{\beta} - 1\right)a_2 y^2 + \cdots\cdots\right] = 0 \quad —(2)$$

This equation can be satisfied only if each power of y is zero. So, eqn (2) reduced to —

for y^0 $\quad 1\times 2a_2 + \left(\frac{\alpha}{\beta} - 1\right)a_0 = 0$

for y^1 $\quad 2\times 3a_3 + \left(\frac{\alpha}{\beta} - 1 - 2\times1\right)a_1 = 0$

P.T.O

for y^2 $\quad 3\times4\, a_4 + (\alpha/\beta - 1 - 2\times2)\, a_2 = 0$

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for y^K $\quad (K+1)(K+2)\, a_{K+2} + (\alpha/\beta - 1 - 2K)\, a_K = 0$

So, a general formula may be derived for the determination of coefficients as:

$$a_{K+2} = \frac{-(\alpha/\beta - 1 - 2K)}{(K+1)(K+2)}\, a_K \quad \text{—— (3)}$$

Where, K is an integer.
This is called Recursion formula.

In the eq^s - (3) as $y \to \infty$, $F \to \infty$ i.e, the power series will consiste of infinite number of terms. So, in order to restrict the number of terms in the series, for a certain value of K (let $K = n$), the numerator in eq^s - (3) vanishes, then

$$\alpha/\beta - 1 - 2K = \alpha/\beta - 1 - 2n = 0$$

or $$\alpha/\beta = 2n+1$$

where, $n = 0, 1, 2$ - - - - - - -

The resulting series with finite number of terms is called Hermite Polynomial $H_n(y)$.
The Hermite Polynomials are defined by

$$H_n(y) = (-1)^n \exp(y^2) \cdot \frac{d^n [\exp(-y^2)]}{dy^n}$$

p.t.o

eg. Hermite Polynomial for $n = 2$

$$H_2(y) = (-1)^2 \exp(y^2) \frac{d^2[\exp(-y^2)]}{dy^2}$$

$$H_2(y) = \exp(y^2) \cdot [-2\exp(-y^2) + 4y^2(\exp(-y^2))]$$

$$H_2(y) = (4y - 2)$$

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