W Conservation Theorems.

Conservation of Energy, Linear and Angular momentum.

It is clear that conservation theorems are only applicable when the space is homogeneous and isotropic and also the time should be homogeneous. Homogeneity and isotropy of space means space remains unaffected with regard to translational and rotational motion. Similarly its time remains unchanged.

Let us now discuss conservation theorems applying Lagrange's equation of motion.

Conservation of Energy

Let us now consider a conservative system for which the forces are derivable from a Potential function V dependent of Position only. The Lagrangian L is given by

$$L = L(q, \dot{q}, t)$$

If L does not depend upon time explicitly, then $L = L(q, \dot{q})$

The total time derivative of L is thus given by

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \quad — ①$$

As we know that the Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

with this substitution eqn. ① takes the form

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k$$

$$= \sum_k \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right)$$

or $$\frac{d}{dt}\left(L - \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

thereby indicating that

$$L - \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = \text{Constant} \quad — ②$$

Now, since $\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V)$

$$= \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \sum_k \left(\frac{1}{2} m_k \dot{q}_k^2\right)$$

$\left\{ \text{because } T = \sum_k \frac{1}{2} m_k \dot{q}_k^2 \text{ and } \frac{\partial V}{\partial \dot{q}_k} = 0 \right\}$

$\therefore \frac{\partial L}{\partial \dot{q}_k} = m_k \dot{q}_k$. Now putting this value in equation ②, we obtain

$$L - \sum_k m_k \dot{q}_k^2 = \text{Const.}$$

$$\text{or } L - 2T = T - V - 2T = \text{Const.}$$

$$= -(T+V) = \text{Const.}$$

Therefore $T + V =$ total energy $=$ Constant

Thus, the energy conservation theorem states that, if the Lagrangian function does not contain the time explicitly, the total energy of the conservative system is conserved.

Conservation theorem for linear momentum

Let us now consider a conservative system for which $\frac{\partial V}{\partial \dot{q}_k} = 0$ and $\frac{\partial T}{\partial q_k} = 0$

The Lagrange's equation for such a system may be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

ie $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial V}{\partial q_k} = 0$$

So $$\dot{P}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = -\frac{\partial V}{\partial q_k} = Q_k \text{ (generalised force)} \quad \text{—(1)}$$

Now if we show that the generalised force Q_k represents the component of total force along the direction of translation of the system and P_k (generalised momentum) is the component of total linear momentum along the same direction then eqn. (1) will be the equation of motion for total linear momentum.

The generalised force Q_k is given by $Q_k = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$. If $\vec{n}$ is the unit vector along the direction of translation of the system, then $\delta \vec{r}_i = \vec{n}\, \delta q_k$ ie $\frac{\partial \vec{r}_i}{\partial q_k} = \vec{n}$

So $$Q_k = \sum_i \vec{F}_i \cdot \vec{n} = \vec{n} \cdot \vec{F} \quad \text{—(2)}$$

which represents the component of total force along the direction of $\vec{n}$ (ie translation)

Again, the kinetic energy $T = \frac{1}{2}\sum_i m_i \dot{r}_i^2$

So the generalised momentum

$$P_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}$$

$$= \sum_i m_i \vec{V}_i \frac{\partial \vec{r}_i}{\partial q_k} \quad \left(\text{Since } \frac{\partial \dot{r}_i}{\partial \dot{q}_k} = \frac{\partial r_i}{\partial q_k}\right)$$

$$= \sum_i m_i \vec{V}_i \cdot \vec{n} = \vec{n} \cdot \sum_i m_i \vec{V}_i$$

which shows that P_k represents the component of total linear momentum along the direction of translation. So, we can say that

$\dot{P}_k = Q_k$ which represents the equation of motion for total linear momentum.

Now, if $Q_k = 0$, $\dot{P}_k = 0$ i.e. P_k = Constant.

This gives the Conservation of linear momentum which states that if a given component of the total applied force vanishes, the corresponding component of linear momentum is conserved.

Conservation theorem for Angular momentum

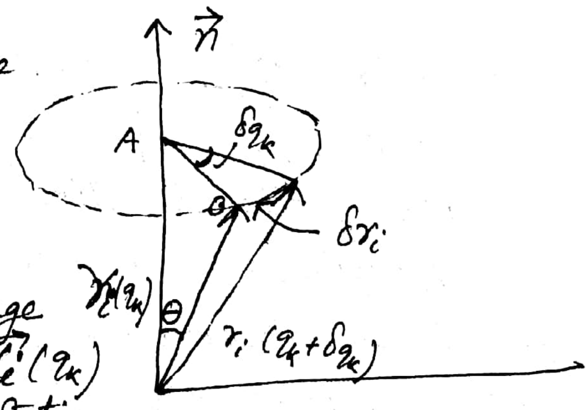
Let us consider a conservative system for which the Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

i.e. $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = -\frac{\partial V}{\partial q_k} = Q_k$$ so that

$$\dot{P}_k = Q_k \quad \text{—} (1)$$

Now, if we show that with rotation coordinate q_k, the generalised force Q_k is the component of total applied torque about the axis of rotation and the generalised momentum P_k is the component of angular momentum along the same axis, the eqn. (1) will represent the equation of motion for the total angular momentum.



The generalised force Q_k is given by

$$Q_k = \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_k}$$

The magnitude of change in position coordinate $\vec{r}_i(q_k)$ due to a change in rotation coordinate q_k is given by

$$|\delta \vec{r}_i| = AB\, \delta q_k$$

$$= r_i \sin\theta\, \delta q_k$$

i.e. $$\left|\frac{\partial \vec{r}_i}{\partial q_k}\right| = r_i \sin\theta \quad \text{—} (2)$$

Scanned with CamScanner

If $\vec{n}$ is the unit vector along the axis of rotation, eqn. (2) may be written as

$$\frac{\partial \vec{r_i}}{\partial q_k} = \vec{n} \times \vec{r_i} \quad \text{—(3)}$$

Now the generalised force can be expressed as

$$Q_k = \sum_i \vec{F_i} \cdot \vec{n} \times \vec{r_i} = \sum_i \vec{n} \cdot (\vec{r_i} \times \vec{F})$$

$$= \sum_i \vec{n} \cdot \vec{N_i} = \vec{n} \sum_i \vec{N_i} = \vec{n} \cdot \vec{N} \quad \text{—(4)}$$

where $\sum_i N_i = \vec{N}$ is the total torque. Thus eqn. (4) shows that Q_k is the component of total torque along the axis of rotation

$$\text{Also } P_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \vec{V_i} \frac{\partial \vec{r_i}}{\partial q_k}$$

$$= \sum_i m_i \vec{V_i} \cdot \vec{n} \times \vec{r_i}$$

$$= \sum_i \vec{n} \cdot \vec{r_i} \times m_i \vec{V_i}$$

$$= \vec{n} \cdot \sum_i \vec{L_i} = \vec{n} \cdot \vec{L} \quad \text{—(5)}$$

where $\sum_i \vec{L_i} = \vec{L}$ is the total angular momentum. Equation (5) shows that P_k is the component of total angular momentum along the axis of rotation.

So we can say that $\dot{P}_k = Q_k$ —(6)

which represents the equation of motion for the total angular momentum of the system.

If q_k is cyclic coordinate, then

$$\vec{n} \cdot \vec{N} = Q_k = -\frac{\partial V}{\partial q_k} = -\frac{\partial L}{\partial q_k} = 0$$

Hence $\dot{P}_k = 0$ i.e. $P_k = \vec{n} \cdot \vec{L} =$ Constant.

Thus the conservation theorem for angular momentum states that if the rotation coordinate q_k is cyclic i.e. if the component of applied torque along the axis of rotation vanishes, then the component of total angular momentum along the axis of rotation is conserved.

Symmetry Properties

The significance of cyclic translation or rotation coordinates in relation to the properties of the system deserves some notice at this point. If a coordinate corresponding to a displacement is cyclic it means that a translational of the system, as if rigid, has no effect on problem. In other words, if the system is invariant under translation along a given direction the corresponding linear momentum is conserved. Similarly the fact that a rotation coordinate is cyclic indicates that the system is invariant under rotation about the given axis. Thus the momentum conservation theorems are closely connected with the symmetry properties of the system. If the system is spherically symmetric we can say without further ado that ~~the~~ all components of angular momentum are conserved.