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PG (Sem - IV)

Plasma physics

Unit - 2

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MAGNETO-HYDRODYNAMICS EQUATION

The fundamental equations of magneto-hydrodynamics rest on the assumption that the plasma may be considered as two interacting charged fluids, one is the electron gas and the other a single type of ion-gas. If such a charged fluid moves in a magnetic field, electric currents are induced in the fluid as a result of its motion. Obviously there must be a magnetic stress $\frac{1}{c}(\vec{i}\times\vec{B})$ in that fluid and hence the gain of momentum in a fluid element i.e the momentum equation of hydrodynamics becomes,

$$\rho\left(\frac{du}{dt}\right) = -\text{grad}\, P + \rho g + \frac{1}{c}(\vec{i}\times\vec{B}) \quad \ldots (1)$$

where ρ = Density ~~and~~, v = velocity and P = Pressure

The set of equations, obtained by truncating the moment equations with the help of various assumptions at any given stage for the solution of any particular problem, is known as "MHD equations".

First, it is assumed that the gravitational and viscous force do not affect the fluid i.e these are neglected in eqⁿ (1), we have

$$\rho\left(\frac{d\vec{v}}{dt}\right) = -\,\text{grad}\,p + \frac{1}{c}(\vec{i}\times\vec{B}) \longrightarrow (2)$$

the mobile operator is defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\,\text{grad} \ldots \ldots \ldots (3)$$

From equation (2) & (3), we get

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\,\text{grad}\,\vec{v}\right] = -\,\text{grad}\,p + \frac{1}{c}(\vec{i}\times\vec{B}) \longrightarrow (4)$$

The 2nd assumption is that, the charge is taken as unit i.e $q = 1$ coulomb and area = unit.
∴ The total electric field E_t is given by

$$\vec{E}_t = \frac{\vec{F}_t}{q=1} = \vec{E} + \frac{v}{c}\times\vec{B} \longrightarrow (5)$$

∴ the electric current

$$J = \frac{\vec{i}}{A=1} = \sigma E_t$$

$$\therefore\ \vec{i} = \sigma\left[\vec{E} + \frac{v}{c}\times\vec{B}\right] \longrightarrow (6)$$

Now from Maxwell's equation of electrodynamics we have,

$$\left.\begin{aligned}
\frac{1}{\mu}\,\text{curl}\,\vec{B} &= \frac{4\pi i}{c} \quad \ldots\ldots (i)\\
\text{curl}\,\vec{E} &= -\frac{1}{c}\frac{d\vec{B}}{dt} \quad \ldots\ldots (ii)\\
&\quad \ldots\ldots (iii)\\
\text{div}\,\vec{B} &= 0 \quad \ldots\ldots (iv)\\
\text{div}\,\vec{E} &= \frac{4\pi e}{\epsilon}
\end{aligned}\right\}(7)$$

From eqⁿ 7(i), we obtain

$$\vec{i} = \frac{c}{\mu}\frac{1}{4\pi}\,\text{curl}\,\vec{B} \longrightarrow (8)$$

$$\therefore \frac{1}{c}(\vec{i}\times\vec{B}) = \frac{1}{c}\left[\frac{c}{\mu}\cdot\frac{1}{4\pi}\,\mathrm{Curl}\,\vec{B}\times\vec{B}\right]$$
$$= \frac{1}{4\pi\mu}\left[\mathrm{Curl}\,\vec{B}\times\vec{B}\right]$$
$$= \frac{1}{4\pi\mu}\left[\vec{B}\cdot(\vec{\nabla}\vec{B}) - \nabla^2\vec{B}\right]$$
$$= \frac{1}{4\pi\mu}\left[\vec{B}\cdot(\vec{\nabla}\vec{B}) - \frac{1}{2}\vec{\nabla}\vec{B}^2\right] \longrightarrow (8)$$

Using eq^n (4) and (8), we have

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\,\mathrm{grad}\,\vec{v}\right] = -\vec{\nabla}\vec{p} - \frac{1}{8\pi\mu}\cdot\vec{\nabla}B^2 + \frac{1}{4\pi\mu}\vec{B}\cdot\vec{\nabla}\vec{B} \longrightarrow (9)$$

Taking curl of eq^n 7(i), we have

$$\mathrm{Curl}\left(\frac{1}{\mu}\,\mathrm{Curl}\,\vec{B}\right) = \frac{4\pi}{c}\,(\mathrm{Curl}\,\vec{i})$$

or, $\frac{1}{\mu}\left[\mathrm{Curl}\,\mathrm{Curl}\,\vec{B}\right] = \frac{4\pi}{c}\,\mathrm{Curl}\left[\sigma\left\{\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right\}\right]$

or, $\frac{1}{\mu}\left[\mathrm{grad}\,\mathrm{div}\,\vec{B} - \nabla^2\vec{B}\right] = \frac{4\pi\sigma}{c}\left[\mathrm{Curl}\,\vec{E} + \mathrm{Curl}\,\frac{\vec{v}\times\vec{B}}{c}\right]$

or, $-\frac{1}{\mu}\nabla^2\vec{B} = \frac{4\pi\sigma}{c^2}\left[c\,\mathrm{Curl}\,\vec{E} + \mathrm{Curl}(\vec{v}\times\vec{B})\right]$ $\quad\left(\because \mathrm{div}\,\vec{B} = 0 \text{ see } (7\,iii)\right)$

or, $\frac{1}{\mu}\nabla^2\vec{B} = \frac{4\pi\sigma}{c^2}\left[\frac{\partial\vec{B}}{\partial t} - \mathrm{Curl}(\vec{v}\times\vec{B})\right]$ (From 7(ii)) $\longrightarrow (10)$

As $\mathrm{Curl}(\vec{v}\times\vec{B}) = \vec{v}\,\mathrm{div}\,\vec{B} - \vec{B}\,\mathrm{div}\,\vec{v} - (\vec{v}\,\mathrm{grad})\vec{B} + (\vec{B}\,\mathrm{grad})\vec{v}$

then $\frac{1}{\mu}\nabla^2\vec{B} = \frac{4\pi\sigma}{c^2}\left[\frac{\partial\vec{B}}{\partial t} - \vec{v}\,\mathrm{div}\,\vec{B} + \vec{B}\,\mathrm{div}\,\vec{v} + (\vec{v}\nabla)\vec{B} - (\vec{B}\nabla)\vec{v}\right]$

$$= \frac{4\pi\sigma}{c^2}\left[\frac{\partial\vec{B}}{\partial t} + \vec{B}\,\mathrm{div}\,\vec{v} + (\vec{v}\nabla)\vec{B} - (\vec{B}\nabla)\vec{v}\right]$$

$$\therefore \frac{c^2}{4\pi\sigma\mu}\nabla^2\vec{B} = \frac{\partial\vec{B}}{\partial t} + \left[\vec{B}\,\mathrm{div}\,\vec{v} + (\vec{v}\nabla)\vec{B} - (\vec{B}\nabla)\vec{v}\right] \longrightarrow (11)$$

Hence eq^n (9) becomes,

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\nabla\vec{v}\right] = -\vec{\nabla}\vec{p} - \frac{1}{8\pi\mu}\cdot\nabla\vec{B}^2 + \frac{1}{4\pi\mu}\vec{B}\cdot\nabla\vec{B} \longrightarrow (12)$$

and from eq^n(10), it is found that

$$\frac{\partial\vec{B}}{\partial t} + \vec{B}\,\mathrm{div}\,\vec{v} + (\vec{v}\vec{\nabla})\vec{B} - (\vec{B}\vec{\nabla})\vec{v} = \frac{c^2}{4\pi\sigma\mu}\nabla^2\vec{B} \longrightarrow (13)$$

Also for normal fluid, conservation law of ~~motion~~ mass gives us

$$\frac{\partial\rho}{\partial t} + \rho\,\mathrm{div}\,\vec{v} + \vec{v}\,\mathrm{grad}\,\rho = 0 \longrightarrow (14)$$

Eq^n (12), (13) and (14) are the equations of magneto-hydrodynamics. Consider the motion ~~for~~ fr at right angle to the unidirectional magnetic field for which $\vec{B}\cdot\nabla\vec{B} = 0$ & $\vec{v}\nabla\vec{v} + \frac{\partial\vec{v}}{\partial t} = 0$. Equation (12) gives us

$$\vec{\nabla}\vec{p} = \frac{1}{8\pi}\nabla\vec{B}^2$$

or, $p = \frac{B^2}{8\pi}$. Similarly we can calculate the energy in magnetic field of