HAMILTON'S CANNONICAL EQUATION OF MOTION

In the formulation of the laws of mechanics with the help of Lagrangian and the equation of motion follows from it, it has been preassumed that the mechanical state of the system is completely defined. There is yet another mode of description, much more powerful & advantageous than the Lagrangian approach in terms of generalised co-ordinates and velocities. In this formulation the state is described in terms of generalised co-ordinates and momenta. The natural eqn which now arises is another approach of mechanics. For getting this new approach, let us recall the Lagrangian of the system as given by

$$L = L(q_k, \dot{q}_k, t) \quad \text{----} (1)$$

and so $$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t} \quad \text{--} (2)$$

and the Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

or $$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \quad \text{-----} (3)$$

with this substitution, eqn (2) becomes

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}$$

ie $$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = \frac{d}{dt}\left\{L - \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right\} \text{—} (4)$$

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But $\frac{\partial L}{\partial \dot{q}_k} = P_k$ = generalised momenta

Thus eqn (4) may be written as

$$\frac{d}{dt}\left\{ L - \sum_k P_k \dot{q}_k \right\} = \frac{\partial L}{\partial t} \quad \text{or} \quad \frac{d}{dt}\left\{ \sum_k P_k \dot{q}_k - L \right\} = \frac{-\partial L}{\partial t} \quad (5)$$

In this eqn $\sum_k P_k \dot{q}_k - L$ is the integral and Corresponds to the energy of the system given as $\sum_k P_k \dot{q}_k - L = H$

Where H is known as Hamiltonian function, or

$$H = \sum_k P_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad ---- \quad (6)$$

The nature of new fn H still remains to be determined. To achieve this we have from eqn (5) that there are no terms in the right which involve the variation in the velocities $\dot{q}_k$. Hence it is seen that H may be written as

$$H = \sum_k P_k \dot{q}_k - L(q_k, \dot{q}_k, t) = H(P_k, q_k, t) \quad - (7)$$

Which is called the Hamiltonian.

The differential of H is given by

$$dH = \sum_k \frac{\partial H}{\partial P_k} dP_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad —— (8)$$

But from eqn (6)

$$dH = \sum_k \dot{q}_k dP_k + \sum_k P_k d\dot{q}_k - dL \quad --- (9)$$

since $L = L(q_k, \dot{q}_k, t)$

$$\therefore dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

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Hence, $dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt$

where $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$

$$= \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \dot{p}_k dq_k - \sum_k p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \quad --- (10)$$

Now Comparing the Co-efficients of dp_k, dq_k and dt in eqn (8) and eqn (10), we get

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad -- (11)$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} \quad - (12)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad - (13)$$

Eqn (11) & (12) are Called Hamilton's eqn or Hamilton's Canonical eqn of motion. These are the required equations of motion in the Variables p & q. The variables p_k & q_k are said to be Canonically Conjugate. Hamilton's equation Constitute 1st 6N order differential eqn. for a system of N particles in this rectangular Co-ordinates in place of 3N Lagrangian eqn of motion of 2nd order.

PHYSICAL SIGNIFICANCE OF Hamiltonian (H)

Since $H = H(p_k, q_k, t)$

Hence $\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t}$

$$= -\sum_k \dot{p}_k \dot{q}_k + \sum_k \dot{p}_k \dot{q}_k + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

But $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, Hence $\frac{dH}{dt}$

~~But ∂H~~ If L is not an explicit fn of time,

$\frac{\partial L}{\partial t} = 0$, thereby giving $\frac{dH}{dt} = 0$ so $H =$ Constant

Thus we may show that if the Lagrangian L is not an explicit fn of time, the Hamiltonian H is constant of motion. This means that it will not appear in H. For conservative system the P.E V is coordinate dependent but velocity independent i.e $\frac{\partial V}{\partial \dot{q}_k} = 0$

Also $H = \sum_k p_k \dot{q}_k - L = 2T - L = 2T - T + V$
$= T + V = E =$ total energy of the system

Thus for conservative system where the co-ordinate transformation is independent of time, the Hamiltonian function H represent the total energy of the system.

Thus H possesses the dimension of energy but in all cases it is not equal to total energy E.

<u>Advantages</u>: Hamiltonian approach is advantageous than Lagrangian approach. Because of the fact that Hamiltonian approach gives deeper insight of the physical problems than Lagrangian. In another way we can say that in case of Hamiltonian approach we have to solve two differential equations of first order than one 2nd order differential eqn in Lagrangian. It is easier to solve first order d.E than to 2nd order d.E. Hence it is superior to Lagrangian approach.