State & Prove Principle of least action

Principle of least Action

Another variational principle associated with the Hamiltonian formulation is known as the Principle of least Action. In mechanics, action is a quantity defined most generally as

$$A = \int_{t_1}^{t_2} 2T\,dt = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k\,dt \quad \text{--- (1)}$$

Therefore the Principle of Least Action states that in a system for which H is conserved

$$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k\,dt = 0 \quad \text{------ (2)}$$

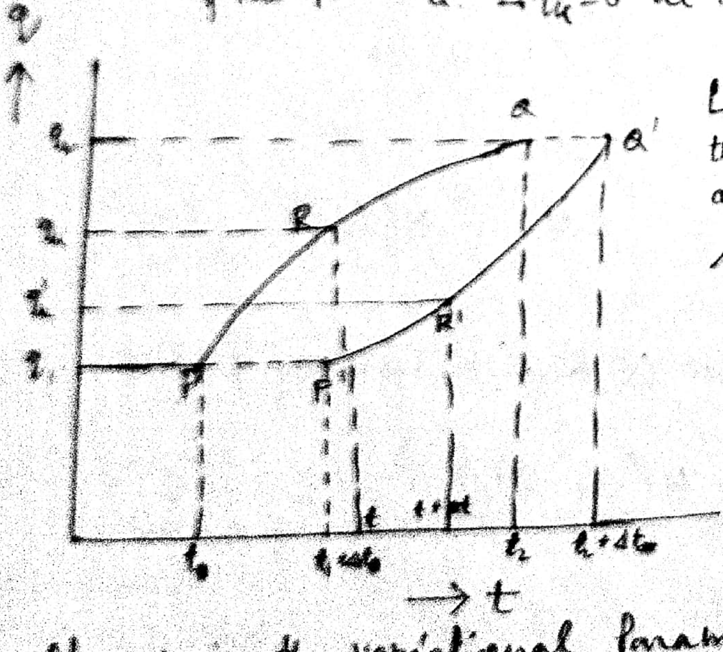
where Δ represents a new type of variation of path which allows time as well as position to vary.

In order to deduce this principle, we have to use Δ variation in which

(i) Time as well as position Co-ordinates are allowed to vary

(ii) Time varies even at end points of the Path

(iii) The position Co-ordinates are held fixed at the end points of the Path i.e. $\Delta q_k = 0$ at the end points.



Let PRQ be the actual Path and P'R'Q', the varied Path. The end points P & Q after time Δt takes the Position P'R'Q' such that position Co-ordinates of P and Q are fixed while the time t is not fixed. A point R on actual Path now goes on R' with the Correspondence

$$q_k \rightarrow q_k' = q_k + \Delta q_k.$$

If α is the variational Parameter, then in δ process t is independent of α but in Δ process t is ~~even~~ function of α even at end points i.e. $t = t(\alpha)$

Thus the function q_k depends upon t and α throughout

i.e. $q_k = q_k(t, \alpha)$

Analytically Δ variation is defined as

$$\Delta q_k = \left[\frac{d}{d\alpha} q_k(\alpha, t)\right] d\alpha = \left[\frac{\partial q_k}{\partial \alpha} + \frac{dq_k}{dt}\frac{dt}{d\alpha}\right] d\alpha$$

$$= \frac{\partial q_k}{\partial \alpha} d\alpha + \dot{q}_k \frac{dt}{d\alpha} d\alpha$$

But $\delta q_k = \frac{\partial q_k}{\partial \alpha} d\alpha$ and $\dot{q}_k \frac{dt}{d\alpha} d\alpha = \dot{q}_k \Delta t$

So $\Delta q_k = \delta q_k + \dot{q}_k \Delta t$ ——— (3)

The relation beetween Δ-variation and δ-variation can now shown to hold for any function $f(q_k, t)$ as

$$\Delta f = \delta f + \dot{f} \Delta t \quad \left[\Delta f = \sum_k \frac{\partial f}{\partial q_k} \Delta q_k + \frac{\partial f}{\partial t} \Delta t \right.$$

$$= \sum_k \frac{\partial f}{\partial q_k} (\delta q_k + \dot{q}_k \Delta t) + \frac{\partial f}{\partial t} \Delta t$$

Thus. $\Delta = \delta + \Delta t \frac{d}{dt}$ —— (4)

$$\left. = \delta f + \dot{f} \Delta t \right]$$

It is to be noted that the Δ operation and time differentiation can't be interchanged in this case which is done in δ-variation

Now from eqn. (1), we have

$$A = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt = \int_{t_1}^{t_2} (L + H)\, dt$$

$$= \int_{t_1}^{t_2} L dt + H(t_2 - t_1) \quad \text{—— (5)}$$

The Δ-variation of eqn. (5) has the following form

$$\Delta A = \Delta \int_{t_1}^{t_2} L dt + H(\Delta t_2 - \Delta t_1) \quad \text{——— (6)}$$

Let us now solve the integral

$$\Delta \int_{t_1}^{t_2} L dt$$

It is also remembered that t_1 & t_2 limits are also subjected to change in this variation, Δ can't be taken inside the integral.

let

$$\int_{t_1}^{t_2} L dt = I$$

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So

$$\Delta I = \delta I + \dot{I}\,\Delta t \quad \text{from eqn (4)}$$

Therefore

$$\Delta\int_{t_1}^{t_2} L\,dt = \Delta I(t_2) - \Delta I(t_1)$$

$$= \left[\delta I(t_2) + \dot{I}(t_2)\,\Delta t_2\right] - \left[\delta I(t_1) + \dot{I}(t_1)\,\Delta t_1\right]$$

$$= \delta I(t_2) - \delta I(t_1) + \dot{I}(t_2)\,\Delta t_2 - \dot{I}(t_1)\,\Delta t_1$$

$$= \delta\int_{t_1}^{t_2} L\,dt + \left[L\,\Delta t\right]_{t_1}^{t_2} \quad \text{—— (7)}$$

From the nature of δ-variation, we have

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\left(\sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right) dt$$

which by Lagrange's equations can be written

$$\delta\int_{t_1}^{t_2} L\,dt = \sum_k \int_{t_1}^{t_2}\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\delta q_k)\right\} dt$$

$$= \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt$$

with substitution equation (7) becomes

$$\Delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt + \left[L\,\Delta t\right]_{t_1}^{t_2} \quad \text{—— (8)}$$

using eqn. (3) to the first integral term of eqn (8)

we have

$$\Delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_k\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\Delta t\right)\right) dt \left[\delta q_k = \Delta q_k - \dot{q}_k\Delta t\right] + \left[L\,\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\,\Delta t\right]_{t_1}^{t_2} + \left[L\,\Delta t\right]_{t_1}^{t_2} \quad \text{—— (9)}$$

At the end points $\Delta q_k = 0$ but Δt does not. So equation (9) becomes

$$\Delta\int_{t_1}^{t_2} L\,dt = \left[-\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\,\Delta t\right]_{t_1}^{t_2} + \left[L\,\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-\sum_k p_k \dot{q}_k\,\Delta t\right]_{t_1}^{t_2} + \left[L\,\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\left(L - \sum_k p_k \dot{q}_k\right)\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-H\,\Delta t\right]_{t_1}^{t_2} \qquad (10)$$

If we restrict to the systems for which H is const, then $\frac{\partial H}{\partial t} = 0$

Thus $\left[H\Delta t\right]_{t_1}^{t_2} = \Delta\int_{t_1}^{t_2} H\,dt$

Substituting this in eq (10), we get $\Delta\int_{t_1}^{t_2} L\,dt = -\Delta\int_{t_1}^{t_2} H\,dt$ *

~~Combining these terms, the total variation of action is~~

$$\cancel{\Delta A = \left[\left(-H - \sum_k p_k \dot{q}_k + L + H\right)\Delta t\right]_{t_1}^{t_2}}$$

$$\cancel{= \left[\left(-H + H\right)\Delta t\right]_{t_1}^{t_2} = 0} \qquad (11)$$

~~This completes the proof of Principle of least action~~

* $$\Delta\int_{t_1}^{t_2} (L+H)\,dt = 0$$

or $$\Delta\int_{t_1}^{t_2}\left(\cancel{L} + \sum_k p_k \dot{q}_k - \cancel{L}\right)dt \Rightarrow$$

$$= \Delta\int_{t_1}^{t_2} \sum p_k \dot{q}_k\,dt = 0 \Rightarrow \Delta\int_{t_1}^{t_2} 2T\,dt = 0 \qquad (11)$$

which is the Principle of least action.

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