THE LAGRANGE EQUATION OF MOTION FROM D' ALEMBERT'S PRINCIPLE

Let us consider a system of particles having 'f' independent generalised co-ordinates to specify the states of its particles.

The transformation equations of the generalised co-ordinates can be given by

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3 \cdots\cdots q_f, t) \quad \text{--- (1)}$$

Let us now suppose that any particle (say ith) of mass m_i be acted upon by an external force $\vec{F}_i$. Then according to D' Alembert's Principle, we have

$$\left.\begin{aligned} &\sum_i (\vec{F}_i - \vec{\dot{P}}_i) \cdot \delta\vec{r}_i = 0 \\ \text{or} \quad &\sum_i (\vec{F}_i - m_i \vec{\ddot{r}}_i) \cdot \delta\vec{r}_i = 0 \end{aligned}\right\} \quad \text{--- (2)}$$

Where $-\vec{\dot{P}}_i$ is the reverse effective force and $\delta\vec{r}_i$ is the virtual displacement of ith particle due to external force $\vec{F}_i$.

Now applying the usual rules of calculus of partial differentiation to eqn (1) we obtain

$$\dot{\underline{r}}_i = \frac{\delta \underline{r}_i}{\delta t} = \underline{v}_i = \frac{\partial r_i}{\partial q_1}\dot{q}_1 + \frac{\partial \underline{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \underline{r}_i}{\partial t}$$

$$= \sum_k \frac{\partial \underline{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \underline{r}_i}{\partial t} \quad \text{--- (3)}$$

Similarly, the arbitrary virtual displacement $\delta \underline{r}_i$ can be connected with virtual displacement in terms of generalised coordinates δq_k by

$$\delta \underline{r}_i = \sum_k \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \quad \text{— (4)}$$

There is no variation of time δt in above eqn. Since a virtual displacement by definition considers no variation of time, but of displacement only.

Equation (2) can now be written in the form as

$$\sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_i \underline{F}_i \cdot \delta \underline{r}_i \quad \text{— (5)}$$

Putting the value of eqn. (4) in eqn. (5), we get

$$\sum_i \sum_k m_i \ddot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k = \sum_i \sum_k \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \quad \text{— (6)}$$

Let us now consider the relation

$$\sum_i m_i \ddot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} = \sum_i \left\{ \frac{d}{dt}\left(m_i \dot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} \right) - m_i \dot{\underline{r}}_i \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \right\}$$

[Since $\frac{d}{dt}\left(\dot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} \right) = \ddot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} + \dot{\underline{r}}_i \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right)$

$$\therefore \ddot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} = \frac{d}{dt}\left(\dot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} \right) - \dot{\underline{r}}_i \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \Big]$$

With the use of above eqn. in eqn. (6), it reduces to . . . .

P. Next Page

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left(\dot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k} \right) - \dot{\underline{r}}_i \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \right] \delta q_k$$

$$= \sum_i \sum_k F_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \quad \text{--- (7)}$$

Differentiating eqn. (3) w.r.t $\dot{q}_k$ partially we obtain

$$\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_k} = \frac{\partial \underline{r}_i}{\partial q_k} \quad \text{--- (8)}$$

Again differentiating eqn. (3) w.r.t q_k partially we get

$$\frac{\partial \dot{\underline{r}}_i}{\partial q_k} = \frac{\partial^2 \underline{r}_i}{\partial q_1 \partial q_k} \dot{q}_1 + \frac{\partial^2 \underline{r}_i}{\partial q_2 \partial q_k} \dot{q}_2 + \cdots\cdots + \frac{\partial^2 \underline{r}_i}{\partial q_k \partial q_k} \dot{q}_k + \frac{\partial^2 \underline{r}_i}{\partial t \partial q_k}$$

$$= \sum_k \frac{\partial^2 \underline{r}_i}{\partial q_k \partial q_k} \dot{q}_k + \frac{\partial^2 \underline{r}_i}{\partial t \partial q_k} \quad \text{--- (9)}$$

Which is analogous to eqn.

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) = \frac{\partial}{\partial q_1}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \dot{q}_2 + \cdots\cdots + \frac{\partial}{\partial q_k}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) \dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right)$$

$$= \sum_k \frac{\partial^2 \underline{r}_i}{\partial q_k \partial q_k} \dot{q}_k + \frac{\partial^2 \underline{r}_i}{\partial t \partial q_k} \quad \text{--- (10)}$$

Therefore, On Comparing eqn. (9) & eqn. (10), we have

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_k} \right) = \frac{\partial \dot{\underline{r}}_i}{\partial q_k} \quad \text{--- (11)}$$

We, then have

$$\frac{d}{dt}\left(\underline{\dot{r}}_i \frac{\partial \underline{\dot{r}}_i}{\partial q_k}\right) = \frac{d}{dt}\left(\underline{\dot{r}}_i \frac{\partial \underline{\dot{r}}_i}{\partial \dot{q}_k}\right) \text{ from eq}^{\underline{n}}\ (8)$$

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{r}_i^2\right)\right\} \quad \text{— } (12)$$

Making substitutions of eq$^{\underline{ns}}$ (11) and (12) in eq$^{\underline{n}}$ (7), we get

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{r}_i^2\right)\right\} - \underline{\dot{r}}_i \frac{\partial \underline{\dot{r}}_i}{\partial q_k}\right] \delta q_k$$

$$= \sum_i \sum_k \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k$$

or $$\sum_i \sum_k \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} m_i \underline{\dot{r}}_i^2\right)\right\} - \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i \underline{\dot{r}}_i^2\right)\right] \delta q_k$$

$$= \sum_i \sum_k \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k$$

or $$\sum_k \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_i \frac{1}{2} m_i \underline{\dot{r}}_i^2\right)\right\} - \frac{\partial}{\partial q_k}\left(\sum_i \frac{1}{2} m_i \underline{\dot{r}}_i^2\right)\right] \delta q_k$$

$$= \sum_i \sum_k \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \quad \text{— } (13)$$

Identifying $\sum_i \frac{1}{2} m_i \underline{\dot{r}}_i^2 = \sum_i \frac{1}{2} m_i \underline{v}_i^2 = T =$ Total Kinetic Energy of the System of Particles

and $\sum_i \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} = Q_k$ where Q_k is called the component of the generalised force. With these substitutions, eq$^{\underline{n}}$ (13) takes the form

$$\sum_k \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right\} \delta q_k = \sum_k Q_k \, \delta q_k$$

$$\text{or } \sum_k \left\{ \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right) - Q_k \right\} \delta q_k = 0 \quad \text{--- (14)}$$

In order to hold the above eqn good, the co-efficient of δq_k must vanish. Hence

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k = 0$$

$$\therefore \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k \quad \text{--- (15)}$$

This is the general form of Lagrange's equation representing second order D.E. There are f such equations corresponding to f generalised co-ordinates. This equation has been derived for a system of particles involving no constraints but in actual problems, there are holonomic constraints mainly.

Hence let us consider the different cases:

Case I when the system is wholly conservative

If the system is wholly conservative, all the forces acting on the system can be derived from a potential function V, given by

$$\underline{F_i} = -\nabla V_i = \frac{-\partial V_i}{\partial \underline{r_i}} \quad \text{where } \nabla$$

acts as differential operator.

Generalised force in this Case Can be written as

$$Q_k = \sum_i \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_k} = -\sum_i \frac{\partial V_i}{\partial \underline{r}_i}\frac{\partial \underline{r}_i}{\partial q_k} = -\sum_i \frac{\partial V_i}{\partial q_k}$$

$$= \frac{\partial}{\partial q_k}\left(\sum_i V_i\right) = -\frac{\partial V}{\partial q_k}$$

where $V = \sum V_i$ = Total Potential energy of the system. Hence the eqn (15) Comes to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(T-V) = 0$$

Since V is the function of position Co-ordinate only and not of generalised velocity $\dot{q}_k$ So the above eqn may be written as

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}(T-V)\right\} - \frac{\partial}{\partial q_k}(T-V) = 0$$

Now putting $L = T-V$, where L is know as Lagrange's function or simply Lagrangian and is defined as the difference between ~~of~~ K.E and P.E of the system.

Therefore the above equation takes the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \quad \text{--- (16)}$$

This is the Lagrangian Equation of motion for holonomic Constraints and Conservative systems. [Note: Case 2 – System is Partially Conservative & Partially non-Conservative
Case – 3 wholly Conservative – are of no use for present Case.]