Variational or Hamilton's Principle

A more general formulation of the Lagrangean in mechanics is due to the variational Principle (so called Hamilton's Principle). The Principle is stated in a generalised form independent of any Co-ordinates system and hence is useful in non-mechanical systems and fields also. It is also known as integral Principle.

This principle may be stated as — "out of all the possible paths along which a dynamical system may move from one point to another point within a given interval of time (consistent with constraints if any), the actual path followed is that which minimizes the time integral of the Lagrangian."

Analytically it can be represented as

$$I = \int_{t_1}^{t_2} L\,dt = \text{extremum} \qquad (1)$$

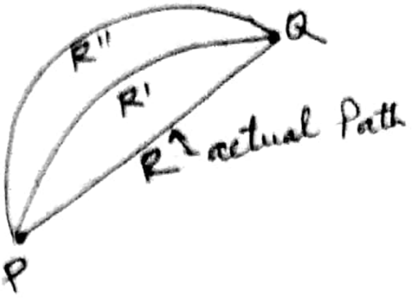
where I is the extremum value of time integral of Lagrangean and is known as Hamilton's principal function of the Path.

Taking δ-variation of the eqn. (1), the variational principle may also be represented as

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \qquad (2)$$

where $L = T - V$, $T = K.E$ & $V = P.E$.

Deduction of Hamilton's Principle



Let us consider that the conservative holonomic dynamical system moves from P to Q, where P & Q are initial and final points at time t_1 & t_2 respectively. Let PRQ be the actual path and PR'Q & PR''Q be the two neighbouring paths out of infinite no. of possibilities.

For the deductions of this principle there must be satisfied two following conditions –

δt must be zero at end points

and δr must be zero at end points.

Let the ith particle of the system be acted upon by a no. of forces given by $\vec{F_i}$ acquiring accleration $\ddot{\underline{r}}_i$, so that we have

$$\vec{F_i} = m_i \ddot{\underline{r}}_i$$

From D' Alembert's Principle, we have

$$\sum_i \left(\vec{F_i} - m_i \ddot{\underline{r}}_i\right) \delta \underline{r}_i = 0$$

$$\text{or} \quad \sum_i \vec{F_i} \cdot \delta \underline{r}_i - \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = 0 \qquad \text{--- (1)}$$

But $\ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}\left(\dot{\underline{r}}_i \cdot \delta \underline{r}_i\right) - \dot{\underline{r}}_i \cdot \frac{d}{dt}\left(\delta \underline{r}_i\right)$ --- (2)

If there is a little variation along the actual and neighbouring paths, we have

$$\frac{d}{dt}(\delta \underline{r}_i) = \delta \frac{d}{dt}(\underline{r}_i) = \delta \dot{\underline{r}}_i \qquad \text{--- (3)}$$

using eqn (3), eqn (2) may be written as

$$\ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}\left(\dot{\underline{r}}_i \cdot \delta \underline{r}_i\right) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \qquad \text{--- (4)}$$

using eqn (4), eqn (1) becomes

$$\sum_i \underline{F}_i\cdot\delta\underline{r}_i - \sum_i m_i\left[\frac{d}{dt}(\dot{\underline{r}}_i\cdot\delta\underline{r}_i) - \dot{\underline{r}}_i\cdot\delta\dot{\underline{r}}_i\right] = 0$$

or $$\sum_i \underline{F}_i\cdot\delta\underline{r}_i - \sum_i m_i\left[\frac{d}{dt}(\dot{\underline{r}}_i\cdot\delta\underline{r}_i) - \frac{1}{2}\delta(\dot{\underline{r}}_i^2)\right] = 0$$

or $$\sum_i \underline{F}_i\cdot\delta\underline{r}_i + \sum_i \frac{1}{2}m_i\,\delta(\dot{\underline{r}}_i^2) = \sum_i \frac{d}{dt}(m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i)$$

or $$\sum_i \underline{F}_i\cdot\delta\underline{r}_i + \delta\sum_i\left(\frac{1}{2}m_i\dot{\underline{r}}_i^2\right) = \sum_i \frac{d}{dt}(m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i) \quad (5)$$

But $\sum_i \underline{F}_i\cdot\delta\underline{r}_i = \delta W$ = work done by the forces $\underline{F}_i$ during displacement $\delta\underline{r}_i$

and $\delta\sum_i\left(\frac{1}{2}m_i\dot{\underline{r}}_i^2\right) = \delta T$ where T is K.E

Therefore eqn. (5) becomes

$$\delta W + \delta T = \sum_i \frac{d}{dt}(m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i)$$

Integrating above eqn between limits t_1 & t_2, we get

$$\int_{t_1}^{t_2}(\delta W + \delta T)\,dt = \int_{t_1}^{t_2}\sum_i \frac{d}{dt}(m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i)\,dt$$

$$= \sum_i\int_{t_1}^{t_2} d(m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i)$$

$$= \sum_i (m_i\dot{\underline{r}}_i\cdot\delta\underline{r}_i)_P^Q = 0$$

Since δr_i at P & Q is zero

For a Conservative System, we know that

$\delta W = -\delta V$ where V is Potential Energy

$$\therefore \int_{t_1}^{t_2}(-\delta V + \delta T)\,dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2}\delta(T-V)\,dt = 0$$

$$\text{or} \quad \delta\int_{t_1}^{t_2} L\,dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} L\,dt = \text{extremum}$$

which is Hamilton's Principle

Scanned with CamScanner