Euler's Equation of Motion of a Rigid body.

In general, a rigid body may be under the influence of a force or a set of forces, which can lead to both translational and Rotational motion. The translatory motion is governed by equation

$$F^{ext} = M\frac{d^2R}{dt^2} = \frac{dP}{dt}$$ where P is the linear momentum

associated with the motion of Centre of mass.

The rotational motion, on the other hand is governed by the equation

$$T = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \frac{dL_i}{dt} = \left(\frac{dL}{dt}\right)_{space}.$$

where $L = \sum_i L_i$

Now in case of rigid body, we have seen that $L_{body} = (I\omega)_{body}$.

The derivative of any physical quantity of a body rotating with angular velocity ω in the space coordinate system is ralated to the derivative in the body coordinate by equation

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$$

In view of this,

$$T_{space} = \left(\frac{dL}{dt}\right)_{space} = \left(\frac{dL}{dt}\right)_{body} + \vec{\omega}\times\vec{L}$$

$$= \frac{d}{dt}(I\omega) + \vec{\omega}\times\vec{L} \quad \text{--- } (1)$$

This is the general equation of motion of a rigid body of any shape rotating about any axis.

For a Symmetric body where the axes of rotation 1, 2 and 3 Coincide with

Principal axis of ~~rotation~~ symmetric, the above eqn. (1) assumes a symmetric form. For rotation about 1-axis of symmetry.

$$\tau_1 = \left| I\dot{\omega}_1 + (\vec{\omega} \times L)_1 \right|$$

$$= \left| I_1\dot{\omega}_1 + [\omega \times (I \cdot \omega)]_1 \right|$$

$$= I_1\dot{\omega}_1 + (\omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2)$$

or $\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$ ——— (2)

Similarly $\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$ ——— (3)

and $\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$ ——— (4)

These equations (2), (3) & (4) are called Euler's equation of motion of a rigid body.

In principle, it should be possible to solve these equations for a symmetric body, to obtain ω_1, ω_2 and ω_3 if τ_1, τ_2 & τ_3 are given.

Special Cases —

(1) <u>For a uniform sphere</u>

$$I_1 = I_2 = I_3 = I$$

Hence $I\dot{\omega}_1 = \tau_1$, $I\dot{\omega}_2 = \tau_2$ and $I\dot{\omega}_3 = \tau_3$

Now the three equations are uncoupled and one can solve these to find the value of ω_1, ω_2 & ω_3 if τ_1, τ_2 & τ_3 are known.

For a special case when $\tau_1 = \tau_2 = \tau_3 = 0$ it is easy to see that

$$I\dot{\omega}_1 = I\dot{\omega}_2 = I\dot{\omega}_3 = 0$$

Hence $\omega_1 = \omega_2 = \omega_3 =$ Constant.

i.e. angular velocity is constant for a torque-free rotation of a sphere.

When the external torque is zero, it is known from conservation of angular momentum that L is constant. So that

$$\left(\frac{dL}{dt}\right)_{body} = 0.$$ This fact, together with $\tau = 0$, used in eqn. (1), gives

$$\vec{\omega} \times \vec{L} = 0$$

This is possible only if $\vec{\omega}$ and $\vec{L}$ are in the same direction & since $L = I\omega$, the quantities $\vec{\omega}$ and $\vec{L}$ being in the same direction implies that I should act as a scalar

(8) One may calculate rate of change of K.E dT/dt from the equation of motion. For this, from eqn. (1) we may write

$$\vec{\omega}\cdot\vec{\tau} = \vec{\omega}\cdot\left(I\frac{d\vec{\omega}}{dt}\right) + \vec{\omega}\cdot(\vec{\omega}\times\vec{L})$$

Since $\vec{\omega}\cdot(\vec{\omega}\times\vec{L}) = (\vec{\omega}\times\vec{\omega})\cdot L = 0$, the above relation becomes

$$\vec{\omega}\cdot\vec{\tau} = \frac{d\vec{\omega}}{dt}\cdot\vec{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}\frac{d}{dt}(\vec{\omega}\cdot\vec{I}\cdot\vec{\omega})$$

But $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\vec{I}\vec{\omega}$

Therefore $\vec{\omega}\cdot\vec{\tau} = \frac{d}{dt}\left(\frac{1}{2}\vec{\omega}\cdot\vec{I}\vec{\omega}\right)$

$$= dT/dt.$$

Gyroscopic Motion →

Rotating Motion of a symmetric top

A body is said to be freely rotating if no torque is acting on it. In practice a torque might have been applied at some time to set the body rotating.

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Then the torque is removed and the body continues rotating freely. In case of free rotating body $T_1 = T_2 = T_3 = 0$ and Euler's equations are modified accordingly. ~~Further~~

We assume that the symmetry of the rotating body is such that $I_1 = I_2 \neq I_3$ which is satisfied by symmetrical tops. Euler's equation then becomes

$$
\left.\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\,\omega_2\omega_3 \\
I_2\dot{\omega}_2 &= I_1\dot{\omega}_2 = (I_3 - I_1)\,\omega_3\omega_1 \\
I_3\dot{\omega}_3 &= 0
\end{aligned}\right\}
\begin{matrix} (5) \\ (6) \\ (7) \end{matrix}
$$

Integration of eqn. (7) shows that ω_3 is const. i.e. the angular velocity around the symmetry axis is constant. Further from eq. (5) & (6), we can write

$$\dot{\omega}_1 = \left\{ \frac{I_1 - I_3}{I_1}\,\omega_3 \right\} \omega_2 = \Omega\,\omega_2 \quad \text{——} (8)$$

Similarly $\dot{\omega}_2 = -\Omega\,\omega_1$ —— (9)

where $\Omega = \dfrac{I_1 - I_3}{I_1}\,\omega_3 = \text{Const.}$ —— (10)

Differentiating eqn. (8) & substituting the value of $\dot{\omega}_2$ from eqn. (9), we get

$$\ddot{\omega}_1 = -\Omega^2 \omega_1 \quad \text{——} (11)$$

This equation is similar to the one obtained for the description of SHM. Therefore, its solution may be written as

$$\omega_1 = A\sin(\Omega t + \theta_0) \quad \text{——} (12)$$

This together with eqn. (9) gives

$$\dot{\omega}_2 = -\Omega A\sin(\Omega t + \theta_0)$$

So that $\omega_2 = A\cos(\Omega t + \theta_0)$

The angular velocities ω_1 and ω_2 are along x and y directions hence the resultant of of ω_1 and ω_2 will always lie in the xy plane. One may write the resultant

$$\omega_p = \omega_1 i + \omega_2 j$$
$$= A \sin(\Omega t + \theta_0)\vec{i} + A \cos(\Omega t + \theta_0)\vec{j}$$

$$\& \quad \omega_p^2 = \omega_1^2 + \omega_2^2 = A^2$$

The total angular velocity ω is given by

$$\omega = \omega_1 \vec{i} + \omega_2 \hat{j} + \omega_3 \vec{k}$$

So that $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$

$$= \omega_p^2 + \omega_3^2 = \text{Const.}$$

Here in such a case the magnitude of the total angular momentum is Const.

The values of A and ω_3 can be expressed in terms of K.E T & angular momentum L as below. we have $I_1 = I_2 \neq I_3$

$$T = \frac{1}{2}\omega \cdot L = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \omega_3^2$$

$$L = I\omega = I_1^2 A^2 + I_3^2 \omega_3^2$$

(using the component form) These equations give

$$\omega_3^2 = \frac{L^2 - 2I_1 T}{I_3(I_3 - I_1)}$$

$$\& \quad A^2 = \frac{L^2 - 2I_3 T}{I_1(I_1 - I_3)}$$

The angle between the vectors $\vec{\omega}$ & $\vec{L}$ is given by

$$\cos\theta = \frac{\vec{\omega}\cdot\vec{L}}{|\vec{\omega}||\vec{L}|} = \frac{\vec{\omega}\,\vec{I}\,\vec{\omega}}{|\vec{\omega}||\vec{L}|}$$

$$= 2T/|\omega||L|$$

This means that, in general $\vec{\omega}$ & $\vec{L}$ are not in the same direction. Further, angle between $\vec{\omega}$ & $\vec{L}$ are is Const. because L is Const. for no external torques.

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