Kepler's Laws of Planetary Motion

On the basis of observations Kepler announced three Laws of planetary motion which are as follows:

(I) All planets revolve in elliptical orbits with the sun at one focus.

(II) The areas swept out by the radius vector from the Sun to a planet in equal times are equal.

(III) The Square of the Period of revolution of any about the sun is proportional to the Cube of t Semimajor axis.

Derivation of Kepler's laws of Planetary motion

1st Law: The first integral of motion under Ce force are

$$J = \mu r^2 \dot{\theta} = \text{Constant} \quad \text{—} (1)$$

and

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{J^2}{2\mu r^2} + V = \text{Const.} \quad (2)$$

From (1),

$$\frac{d\theta}{dt} = \frac{J}{\mu r^2} \quad \text{—} (3)$$

from (2),

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - \frac{J^2}{2\mu r^2} - V\right)} \quad \text{—} (4)$$

Dividing eqn (4) by (3), We get

$$\frac{dr}{d\theta} = \frac{\mu r^2}{J}\sqrt{\frac{2}{\mu}\left(E - \frac{J^2}{2\mu r^2} - V\right)}$$

From eqn (a)

or $d\theta = \dfrac{J\,dr}{\mu r^2\sqrt{\left\{\frac{2}{\mu}\left(E-V-\frac{J^2}{2\mu r^2}\right)\right\}}}$ — (5)

Under inverse square law of force, we have

$F(r) = -\frac{K}{r^2}$ where K is a const.

But $F(r) = -\frac{\partial V}{\partial r}$ $\therefore$ $-\frac{\partial V}{\partial r} = -\frac{K}{r^2}$

or Potential Energy $V = -\frac{K}{r}$ — (6)

Substituting this value of V in eqn (5) we get

$$d\theta = \frac{J\,dr}{\mu r^2\sqrt{\left\{\frac{2}{\mu}\left(E+\frac{K}{r}-\frac{J^2}{2\mu r^2}\right)\right\}}}$$

Integrating, we get

$$\theta = \int \frac{J\,dr}{\mu r^2\sqrt{\left\{\frac{2}{\mu}\left(E+\frac{K}{r}-\frac{J^2}{2\mu r^2}\right)\right\}}} + \theta'$$

where θ' is the const. of integration.

substituting $r = \frac{1}{u}$, we get

$$\theta = -\int \frac{J\,du}{\mu\sqrt{\left\{\frac{2}{\mu}\left(E+Ku-\frac{J^2u^2}{2\mu}\right)\right\}}} + \theta'$$

$$= \theta' - \int \frac{du}{\sqrt{\left(\frac{2\mu E}{J^2}+\frac{2\mu K E}{J^2}-u^2\right)}} = \theta' - \int \frac{du}{\sqrt{\left[\frac{2\mu E}{J^2}+\frac{\mu^2K^2}{J^4}\right]^2 - \left(u-\frac{\mu K}{J^2}\right)}}$$

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$$\theta = \theta' - \cos^{-1} \frac{u - \frac{\mu k}{J^2}}{\sqrt{\left(\frac{2\mu E}{J^2} + \frac{\mu^2 k^2}{J^4}\right)}}$$

which gives

$$\frac{\frac{uJ^2}{\mu k} - 1}{\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1\right)}} = \cos(\theta - \theta')$$

or $$\frac{uJ^2}{\mu k} - 1 = \sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1\right)} \cos(\theta - \theta')$$

or $$u = \frac{\mu k}{J^2}\left[1 + \sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1\right)} \cos(\theta - \theta')\right] \quad -(7)$$

$$= c\left[1 + \epsilon \cos(\theta - \theta')\right]$$ where $c = \frac{\mu k}{J^2}$ & $\epsilon = \sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1\right)}$ —

ie $$\frac{1}{r} = c\left[1 + \epsilon \cos(\theta - \theta')\right] \quad -(9)$$

which is the equation of the conic with ϵ as eccen and one focus as the origin.

Thus the equation of the path of the two-body problem reduced mass μ is always a conic section, which is the generalisation of Kepler's first law.

The nature of conic section depends upon the value of eccentricity given by eqn (8)

The nature of conic section depends upon eccentricity ϵ
if $\epsilon > 1$ or $E > 0$, the conic is hyperbola
if $\epsilon = 1$ or $E = 0$, " " " Parabola
if $\epsilon < 1$ or $E < 0$, conic is an ellipse.
if $\epsilon = 0$, or $E = \frac{\mu k^2}{2J^2}$, the conic is a circle

Second Law Again from $\bar{J} = \mu r^2 \dot{\theta} = \text{const}$.

we see that $\frac{d}{dt}(r^2 \dot{\theta}) = 0$ since μ is const.

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} r^2 \dot{\theta}\right) = 0$ So $\frac{1}{2} r^2 \dot{\theta} = \text{const}$.

where $\frac{1}{2} r^2 \dot{\theta} = \frac{dA}{dt}$ = Areal velocity $= \frac{1}{2} r^2 \frac{d\theta}{dt}$.

$= \frac{1}{2} r . r d\theta$ where $\frac{1}{2} r . r d\theta = dA$

$\therefore$ The area swept out by the radius vector per unit time; is const which is the 2nd law. $\frac{dA}{dt} = \text{const}$

Third Law: If T is the period of time then the area of the orbit is given by

$$A = \int_0^T \frac{dA}{dt}\, dt = \int_0^T \frac{1}{2} r^2 \dot{\theta}\, dt = \int_0^T \frac{J}{2\mu}\, dt = \frac{JT}{2\mu} \quad (10)$$

But the area of the ellipse $A = \pi ab$ where a & b are semi major & semi-minor axes.

$b = a\sqrt{1-\epsilon^2} = a\sqrt{\left(1 - 1 - \frac{2EJ^2}{\mu k^2}\right)} = \sqrt{\left(-\frac{2EJ^2}{\mu k^2}\right)}$ But $E = -\frac{k}{2a}$

therefore $b = a\sqrt{\left(\frac{kJ^2}{a\mu k^2}\right)}$

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$$= a^{\frac{1}{2}}\sqrt{\left(\frac{J^2}{\mu k}\right)}$$

using the value of b, the area of the ellipse

$$A = \pi a b = \pi a^{3/2}\sqrt{\left(\frac{J^2}{\mu k}\right)} \quad \text{—— (11)}$$

Comparing equation (10) & (11), we get

$$\frac{JT}{2\mu} = \pi a^{3/2}\sqrt{\left(\frac{J^2}{\mu k}\right)}$$

squaring, $\frac{J^2 T^2}{4\mu^2} = \pi^2 a^3 \frac{J^2}{\mu k}$

or $T^2 = 4\pi^2 a^3 \frac{\mu}{K}$

So that $T^2 \propto a^3$

ie. the square of the Period of revolution of the Planet around the Sun is proportional to the Cube of the semi-major axis, which is Kepler's third law..

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