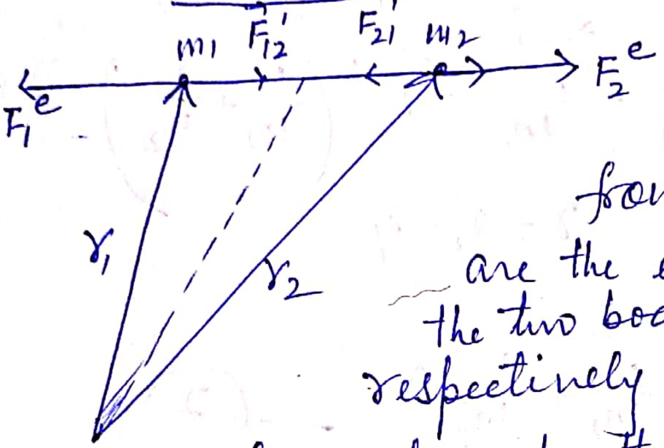
Two-Body Central force Problem
- Reduction to one body Problem

The forces exerted by two bodies on each other form the action and reaction pair and are equal and opposite according to Newton's third law. Under mutual interaction, the two bodies move in such a way that their centre of mass remains fixed in space.

Reduction of two-body Problem to equivalent one body Problem:



Let us consider two bodies of masses m_1 & m_2 seperated by a distance r from each other. F_1^e and F_2^e are the external forces acting on the two bodies of masses m_1 & m_2 respectively. If F_1' & F_2' are the internal forces due to their mutual interaction (say gravitation force)

The equations of motion of the two bodies are given by

$$m_1\ddot{\underline{r}}_1 = \vec{F}_{12}^{\,i} + \vec{F}_1^{\,e} \quad \text{—} (1)$$

$$m_2\ddot{\underline{r}}_2 = \vec{F}_{21}^{\,i} + \vec{F}_2^{\,e} \quad \text{—} (2)$$

Deviding eqn (1) by m_1 and eqn (2) by m_2, we get

$$\ddot{\underline{r}}_1 = \frac{\vec{F}_{12}^{\,i}}{m_1} + \frac{\vec{F}_1^{\,e}}{m_1} \quad \text{—} (3)$$

$$\& \quad \ddot{\underline{r}}_2 = \frac{\vec{F}_{21}^{\,i}}{m_2} + \frac{\vec{F}_2^{\,e}}{m_2} \quad \text{—} (4)$$

Subtracting eqn (3) from eqn (4), we get

$$\ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = \frac{\vec{F}_{21}^{\,i}}{m_2} + \frac{\vec{F}_2^{\,e}}{m_2} - \frac{\vec{F}_{12}^{\,i}}{m_1} - \frac{\vec{F}_1^{\,e}}{m_1} \quad \text{—} (5)$$

But $\frac{F_1^e}{m_1} = \frac{F_2^e}{m_2}$ (As we have already taken in the begining)

Therefore eqn. 5 becomes

$$\ddot{r}_2 - \ddot{r}_1 = \frac{\vec{F}_{21}^{\,i}}{m_2} - \frac{\vec{F}_{12}^{\,i}}{m_1}$$

But according to Newton's third law, the internal force $\vec{F}_{12}^{\,i} = -\vec{F}_{21}^{\,i} = -\vec{F}^{\,i}$ (say)

$$\therefore \ddot{r}_2 - \ddot{r}_1 = \frac{\vec{F}^{\,i}}{m_2} + \frac{\vec{F}^{\,i}}{m_1} = \vec{F}^{\,i}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{—(6)}$$

But from fig we have $r = r_2 - r_1$

Eqn 6 gives

$$\ddot{r} = \vec{F}^{\,i}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{—(7)}$$

Putting $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$ —(8) where μ is called Reduced Mass of the system.

Eqn. 7 Can be written as

$$\vec{F}^{\,i} = \mu\ddot{r} \quad \text{—(9)}$$

According Newton's law of gravitation

$$\vec{F}^{\,i} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Thus eqn. 8 becomes

$$\mu\ddot{r} = -\frac{Gm_1m_2}{r^2}\hat{r} \quad \text{—(10)}$$

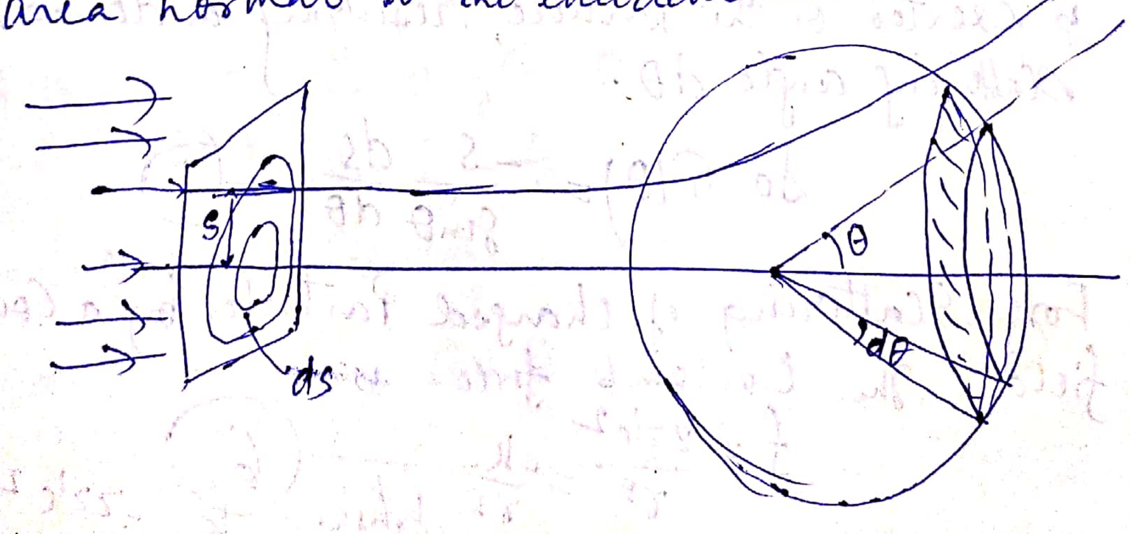
which is one-body problem.

Scattering in a Central force field.

The Scattering Problem is Concerned with the Scattering of Particles by a Centre of force. let us Consider a uniform beam of Particles whether electrons, or α-Particles, or Planets - - - all of the same mass and energy be incident upon a Centre of force. As a Particle approaches the Centre of force it will either be attracted or repelled, and its orbit will deviate from the incident straight line trajectory. After Passing the Centre of force, the force acting on the Particle will eventually diminish so that the orbit once again approaches a st. line. In general the final direction of motion is different from incident direction, and the Particle is said to be scattered. The Cross section for scattering in a given direction $\sigma(\omega)$, is defined by

$$\sigma(\omega)d\omega = \frac{\text{No. of Particles scattered into solid angle } d\omega \text{ per unit time}}{\text{Incident intensity}} \quad \text{—} (1)$$

where $d\omega$ is the element of solid angle in the direction of ω and $\sigma(\omega)$ is often known as differential scattering Cross-section. The incident intensity (I) or flux is defined as the no. of Particles crossing unit area normal to the incident beam in unit time.



With Central forces there must be complete symmetry around the axis of the incident beam, hence the element of solid angle can be written as

$$d\omega = 2\pi \sin\theta \, d\theta \quad \text{--- (2)}$$

where θ is the Scattering angle

The amount of scattering are determined by its energy and angular momentum. It is convenient to express the angular momentum in terms of the energy and impact-Parameter s (defined as the perpendicular distance between the Centre of force and the incident velocity. If v_0 is the incident speed of the particle, then

$$l = m v_0 s = s\sqrt{2mE} \quad \text{--- (3)}$$

But the no. of Particles Scattered into solid angle $d\omega$ lying between θ & $\theta + d\theta$ must be equal to the no. of incident Particles with impact Parameter lying between Corresponding s & $s+ds$

i.e. $$2\pi I s\,ds = -2\pi \sigma(\theta) I \sin\theta \, d\theta \quad \text{--- (4)}$$

The -ve sign is taken because an increase ds in the impact Parameter means less force is exerted on the Particle, resulting in decrease in scattering angle $d\theta$.

$$\text{So } \sigma(\theta) = \frac{-s}{\sin\theta}\frac{ds}{d\theta} \quad \text{--- (5)}$$

For Scattering of charged Particles by a Coulomb field, The Coulomb force is

$$f = \frac{zz'e^2}{r^2} = \frac{-k}{r^2} \quad \text{--- (6)} \quad \text{where } k = -zz'e^2$$

The scattering force field is produced by fixed charge $-Ze$ acting on the incident particle having a charge $-Z'e$

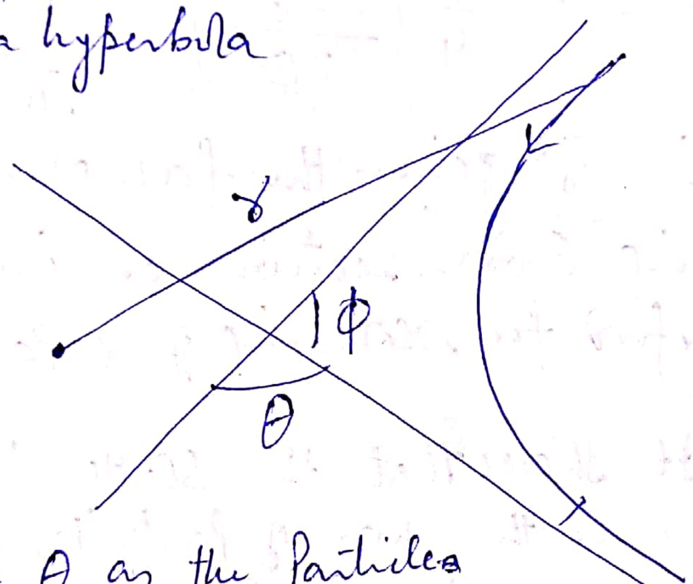
Then the equation for the orbit is simply

$$\frac{1}{r} = -\frac{mZZ'e^2}{l^2}(1+\epsilon\cos\theta) \quad —(7)$$

Where the co-ordinates have been rotated so that $\theta'=0$, and where ϵ is

$$\epsilon = \sqrt{1+\frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1+\left(\frac{2ES}{ZZ'e^2}\right)^2} \quad —(8)$$

From eqn (8), it is obvious that $\epsilon>1$. Hence eq (7) represents a hyperbola



The change in θ as the particles comes in from infinity, is scattered and goes out again to infinity, is clearly the same as the angle between the assymptotes ϕ which is turn the supplement of the scattering angle Θ. So Θ is given by

$$\cot\frac{\Phi}{2} = \sin\frac{\theta}{2} = \frac{1}{\epsilon}$$

or $\cot^2\frac{\Phi}{2} = \operatorname{cosec}^2\Phi/_2 - 1 = \epsilon^2-1$

and finally $\cot\frac{\Theta}{2} = \frac{2ES}{ZZ'e^2} \quad —(9)$

with this result the differential equation can be easily obtained.

From eqn (9)

$$S = \frac{ZZ'e^2}{2E} \cot \theta/2$$

So that $\sigma(\theta)$ becomes (from eqn 5)

$$\sigma(\theta) = \frac{1}{2}\left(\frac{ZZ'e^2}{2E}\right)^2 \frac{\cot \theta/2}{\sin\theta} \cdot \frac{1}{\sin^2 \theta/2}$$

or $$\sigma(\theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \frac{1}{\sin^4 \theta/2} \qquad \text{--- (10)}$$

Equation (10) gives the famous Rutherford scattering cross-section originally derived by Rutherford for scattering of α particle by atomic nuclei.

It shows that the scattering cross section and hence the no. of particles scattered must be proportional to (i) $\operatorname{cosec}^4 \theta/2$

(ii) the square of the nuclear charge (Ze)

(iii) the square of the charge on the particle $(Z'e)$

and (iv) inversely proportional to the square of the initial energy E