Birge-Vieta Method :-

This method is used for finding the real roots of a polynomial equation. This method is based on Newton-Raphson method. Let a polynomial equation for degree n, say

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots a_1 x + a_0 = 0 \quad \text{--- (1)}$$

Let x_0 be an initial approximation to the root α. The Newton-Raphson iterated formula for improving this approximation is

$$x_i = x_{i-1} - \frac{P_n(x_{i-1})}{P_n'(x_{i-1})}, \quad i = 1, 2, \ldots\ldots \quad (2)$$

To apply this formula we should be able to evaluate both $P_n(x)$ and $P_n'(x_i)$ at any x_i. The most natural way is to evaluate.

$$P_n(x_i) = a_n x_i^n + a_{n-1} x_i^{n-1} + \cdots\cdots a_1 x_i + a_0$$

$$P_n'(x_i) = n a_n x_i^{n-1} + (n-1) a_{n-1} x_i^{n-2} + \cdots\cdots 2 a_2 x_i + a_1$$

Thus there is a need to look for some efficient method for evaluating $P_n(x)$ and $P_n'(x)$. Let us consider the evaluation of $P_n(x)$ and $P_n'(x)$ using Horner's method as discussed in the previous section.

We have

$$P_n(x) = (x - x_0)\, q_{n-1}(x) + r_0 \quad \text{--- (3)}$$

where $q_{n-1}(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots\cdots b_2 x + b_1$ — (4)

and $b_0 = P_n(x_0) = r_0$ — (5)

Next we shall find the derivative $P_n'(x_0)$ using Horner's method. We divide $q_{n-1}(x)$ by $(x-x_0)$ using Horner's method. Then, we write

$$q_{n-1}(x) = (x-x_0)\, q_{n-2}(x) + r_1$$

$$q_{n-1}(x) = c_n x^{n-2} + c_{n-1} x^{n-3} + \cdots\ c_3 x + c_2 \text{ --- } ⑥$$

Comparing the coefficients from Eqn ④ & ⑥, we get c_1 as

	b_n	b_{n-1}	---	b_k	---	b_2	--	b_1
x_0	$\longrightarrow$	$x_0 c_n$		$x_0 c_{k+1}$	--	$x_0 c_3$		$x_0 c_2$
	$c_n = b_n$	c_{n-1}	---	c_k	---	c_2	--	c_1

We have,

$$c_1 = q_{n-1}(x_0) \text{ ——— } ⑦$$

Now equations ③ and ⑤

$$P_n(x) = (x-x_0)\, q_{n-1}(x) + P_n(x_0) \text{ ——— } ⑧$$

Differentiating both sides of Eqn ⑧ w.r.t x

$$P_n'(x) = q_{n-1}(x) + (x-x_0)\, q_{n-1}'(x) \text{ ——— } ⑨$$

Putting $x = x_0$ in Eqn ⑨, we get

$$P_n'(x_0) = q_{n-1}(x_0) \text{ ——— } (10)$$

Comparing ⑦ & ⑩

$$P_n'(x_0) = q_{n-1}(x_0) = c_1$$

Hence, the Newton-Raphson method (Eqn 2) simplies to

$$x_i = x_{i-1} - \frac{b_0}{c_1} \qquad \text{—————} \qquad (11)$$

We summarise the evaluation of b_i and c_i in the following table.

	a_n	a_{n-1}	------	a_k	---	a_2	a_1	a_0
x_0		$x_0 b_n$	------	$x_0 b_{k+1}$	...	$x_0 b_3$	$x_0 b_2$	$x_0 b_1$
	$a_n = b_n$	b_{n-1}	------	b_k	----	b_2	b_1	$b_0 = P_n(x_0)$
x_0		$x_0 c_n$		$x_0 c_{k+1}$	----	$x_0 c_3$	$x_0 c_2$	
	$c_n = b_n$	c_{n-1}		c_k		c_2		$c_1 = P_n'(x_0)$

Consider the polynomial $p(x)$ as given in Eqn

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots a_1 x + a_0 \quad \text{—} \ (1)$$

Dividing $p(x)$ by $(x-\alpha)$ we get

$$p(x) = q_0(x)(x-\alpha) + r_0 \quad \text{—————} \ (2)$$

Where $q_0(x)$ is a polynomial of degree $(n-1)$ and r_0 is a constant.

Let $q_0(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots - b_2 x + b_1$

We are denoting the coefficients by $b_1, b_2 \cdots b_n$ instead of $b_0, b_1, \cdots - b_{n-1}$. Set $b_0 = r_0$. Substitute the expressions for $q_0(x)$ and r_0 in Eqn (3) we get.

$$P(x) = (x-\alpha)(b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots b_2 x + b_1) + b_0 \quad \text{--- (3)}$$

$$P(\alpha) = b_0$$

Compare the coefficients, we get (from 1 & 3)

Coefficient of x^n : $a_n = b_n$, $b_n = a_n$

Coefficient of x^{n-1} : $a_{n-1} = b_{n-1} - \alpha b_n$

$b_{n-1} = a_{n-1} + \alpha b_n$

Coefficient of x^k : $a_k = b_k - \alpha b_{k+1}$

$b_k = a_k + \alpha b_{k+1}$

Coefficient of x_0 : $a_0 = b_0 - \alpha b_1$

$b_0 = a_0 + \alpha b_1$

$$\boxed{b_k = a_k + \alpha b_{k+1}}$$

Horner's Table

α	a_n	a_{n-1}	a_{n-2} ...	a_k ...		a_1	a_0
		αb_n	αb_{n-1} ...	αb_{k+1} ...		αb_2	αb_1
	b_n	b_{n-1}	b_{n-2} ...	b_k ...		b_1	$b_0 = P(\alpha)$

We shall illustrate this procedure in an example.