If S & T are two non-empty sets, then mapping from S to T is a rule. ($f : S \to T$)

→ The **Cartesian product** of two sets S and T is defined as $S \times T = \{(a,b) : a \in S, b \in T\}$ → product of sets

We say that $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ & $b_1 = b_2$

Binary Composition '$*$' means addition (+) and multiplication (·) of the elements of two sets.
$a \in S$ & $b \in T$.
e.g. $a * b$ ($a+b$ and $a \cdot b$)

Definition. A mapping (function) $* : S \times S \to S$ is called a binary composition on the sets S, where
$S \times S = \{(a,b) : a, b \in S\}$.
The image of an element $(a,b) \in S \times S$ under $*$ binary composition is usually written as $a * b$ (i.e. $a+b$ & $a \cdot b$)

Note. If $*$ is a binary composition on a sets then $a * b \in S$ for all $a, b \in S$.

Exp. Addition and multiplication are binary composition in the set N of natural numbers $N = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$
since $a+b \in N$ and $a \cdot b \in N$ $\forall a, b \in N$. However, subtraction is not a binary composition in N.
since $2 \in N$ and $3 \in N$

Addition: $2 + 3 = 5 \in N$
multiplication: $2 \cdot 3 = 6 \in N$
} binary composition

but subtraction $2 - 3 = -1 \notin N$ ⇒ Not binary composition

Group:- A non-empty set G with a binary $(+, \cdot)$ composition '$*$' is called a group, if the following conditions are satisfied:-

(1) Closure law:- $a*b \in G, \ \forall a,b \in G$

(A1) $a+b \in G \quad \forall a \in G, b \in G$

(M1) $a \cdot b \in G \quad \forall a \in G, b \in G$

(2) Associative law:- $(a*b)*c = a*(b*c) \ \forall a,b,c \in G$

(A2) $(a+b)+c = a+(b+c) \quad a,b,c \in G$

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad a,b,c \in G$

(3) Commutetive law: $a*b = b*a \quad \forall a,b \in G$

(A3) $a+b = b+a \quad \forall a,b \in G$

(M3) $a \cdot b = b \cdot a \quad \forall a,b \in G$

(4) Existence of Identity:- $\exists e \in G : a*e = e*a = a$

(A4) Addition identity $e=0$, $a+0 = 0+a = a \quad \forall a \in G$

(M4) multifalication identity $e=1$, $a \cdot 1 = 1 \cdot a = a \ \forall a \in G$

(5) Existence of inverse:- $\forall a \in G, \exists b \in G$

$$a*b = b*a = e$$

(A5) Addition inverse $(b=-a)$, $a+(-a) = (-a)+a = 0$

(M5) Multiplication inverse $(b = a^{-1})$ $a \cdot a^{-1} = a^{-1}a = 1$

A group $(G,*)$ is called an abelian (Commutative) group if

$a*b = b*a \quad \forall a,b \in G$

$a+b = b+a \quad \forall a,b \in G$

$a \cdot b = b \cdot a \quad \forall a,b \in G$

Field

Field – A non-empty set F with two binary compositions $(+, \cdot)$ denoted by $+$ and $\cdot$ is called a field, if the following these properties are satisfied.

A 1) $a+b \in F \quad \forall a, b \in F$ Closure law

A 2) $a+b = b+a \quad \forall a, b \in F$ Commutative law

A 3) $a+(b+c) = (a+b)+c \quad \forall a, b, c \in F$ Associative law

A (4) There exists an element $0 \in F$ such that $a+0 = 0+a = a$ $\forall a \in F$ (Additive identity)

A (5) $\forall a \in F$, there exists some element $-a$ in F i.e. $-a \in F$ such that $a+(-a) = 0 = (-a)+a$ (additive inverse)

M (6) $a \cdot b \in F \quad \forall a, b \in F$

M (7) $a \cdot b = b \cdot a \quad \forall a, b \in F$

M (8) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in F$

M (9) $\exists\, e \in F : a \cdot e = e \cdot a = a \quad \forall a \in F$, e is called multipli-cative identity of F. As e by 1, so that $a \cdot 1 = 1 \cdot a = a \quad \forall a \in F$

M (10) $\forall a \in F$ & $a \neq 0$, $\exists\, b \neq 0 \in F$ such that $a \cdot b = b \cdot a = 1$. We write b as a^{-1} and all in the multiplicative inverse of a. Thus $a \cdot a^{-1} = a^{-1} a = 1$

(11) $a \cdot (b+c) = a b + a \cdot c \quad \forall a, b, c \in F$ (Distributive law)

Exp. The set $\mathbb{Q}$ of all rational numbers is a field w.r.t. usual addition and multiplication.