Bilinear form

Definition $\rightarrow$ Let V be a finite dimensional vector space over a field F.

A bilinear form on V is a function $f: V \times V \rightarrow F$ such that

(i) $f(u+v, w) = f(u,w) + f(v,w)$

(ii) $f(u, v+w) = f(u,v) + f(u,w)$

(iii) $f(u, \alpha v) = f(\alpha u, v) = \alpha f(u,v)$

* A mapping $f: V \times V \rightarrow F$ is called a

$e_3 e_2)+$ bilinear form (or bilinear function) on V if the following are satisfied.

Imp. $\Longrightarrow$

(1) $f(u, \alpha v + \beta w) = \alpha f(u,v) + \beta f(u,w)$

(2) $f(\alpha u + \beta v, w) = \alpha f(u,w) + \beta f(v,w)$

where $f(u,w)$ and $f(u,v)$ are linear w.r.t to first variable.

Take $V = \mathbb{R}$ then $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x,y) = xy$

Notice here the difference between linear and bilinear

$f(x,y) = x + y$ is linear

$f(x,y) = xy$ is bilinear

More generally $f(x,y) = \alpha xy$ is bilinear for any $\alpha \in \mathbb{R}$

Exp. (i) The zero function from $V \times V$ into F is a bilinear on V.

(ii) Let g and h be linear functionals on a vector space $V(F)$ s.t. $f : V \times V \to F$ by the formula

$$f(u, v) = g(u) \cdot h(v)$$

Then f is a bilinear form on V. for g and h both are linear map on V into F.

(iii) Define a map $f : R^n \times R^n \to R$ by the formula

$$f(u, v) = \sum_{i=1}^{n} (\alpha_i \cdot \beta_i)$$

$$= \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots \alpha_n\beta_n$$

where $u = (\alpha_1, \alpha_2, \cdots \alpha_n)$ and $v = \{\beta_1, \beta_2, \cdots \beta_n)$

Then f is a bilinear form on R^n.

Note. the set of all bilinear forms on V is denoted by $B(V)$

Let $f, g \in B$ and $\alpha \in F$. we define

$$(f+g)(u, v) = f(u, v) + g(u, v)$$

$$(\alpha f)(u, v) = \alpha f(u, v)$$

It is easy to verify that $B(V)$ is a vector space over k.