Basis of a Vector Space :-

Let V be a vector space over a field F and let S be a any non-empty subset of V, then S is said to be a basis of V if

(i) S is linearly independent ($\Sigma \alpha_i S_i = 0$ $\alpha_i \in F$ $S_i \in V, \alpha_i = 0$)
S generates V i.e
(ii) L(S) = V i.e every element of V is a linear combination of finite elements of S.

Exp. $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ forms a basis of $V(R^3)$, and is called usual basis.

$V(R^3) \rightarrow$ read as :- vector space V over real numbers (R) in 3-dimensions.

→ The zero space has no basis

→ Every finitely generated vector space has a basis.

→ Every non-zero vector space has a basis

→ A vector space may have more than one basis.

Finite Dimensional Vector Space :-

Let V(F) be a vector space over a field F and let S be a non-empty subset of V, then V(F) is said to be finite dimensional if S is finite subset of V such that L(S) = V.

If this set contains n elements, then the dimension of V is n.

General Theorem

Theorem 1- If $S = \{\overset{v_1}{S_1}, \overset{v_2}{S_2}, \overset{v_3}{S_3} \cdots \overset{v_n}{S_n}\}$ is the basis of a vector space $V(F)$, then each element of V is uniquely expressible/ represents as a linear combination of element of S.

Proof: Since $V = L(S)$, each $v \in V$ is expressible as Let two different sets $(\alpha_1, \alpha_2 \cdots \alpha_n)$ and $(\beta_1, \beta_2 \cdots \beta_n)$ be scalars corresponding to an element $s \in V$ such that

$$S = \alpha_1 S_1 + \alpha_2 S_2 + \cdots \alpha_n S_n \text{ and } S = \beta_1 S_1 + \beta_2 S_2 + \cdots + \beta_n S_n \quad ①$$

$$\alpha_1 S_1 + \alpha_2 S_2 + \cdots \alpha_n S_n = \beta_1 S_1 + \beta_2 S_2 + \cdots \beta_n S_n$$

$$\alpha_1 S_1 - \beta_1 S_1 + \alpha_2 S_2 - \beta_2 S_2 + \cdots \alpha_n S_n - \beta_n S_n = 0$$

$$(\alpha_1 - \beta_1) S_1 + (\alpha_2 - \beta_2) S_2 + \cdots + (\alpha_n - \beta_n) S_n = 0$$

Since set $S = (v_1, v_2, \cdots v_n)$ is Linear independent so that

$$\alpha_1 - \beta_1 = 0, \quad \alpha_2 - \beta_2 = 0, \cdots \alpha_n - \beta_n = 0$$

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2, \cdots \alpha_n = \beta_n$$

Hence, the expression ① is unique.

Theorem 2:- (Existence Theorem): Every finitely generated vector space has a finite basis.

Theorem 3:- If V is a finite dimensional vector space and if $S_1, S_2, S_3, \cdots S_m$ spann V, then some subsets of $S_1, S_2, \cdots S_m$ forms a basis of V.

Theorem 4:- If $(S_1, S_2, \cdots S_n)$ is a basis of V(F) and $(V_1, V_2, V_3, \cdots V_m) \subseteq V$, are Linearly independent over F, then $m \leq n$.

Theorem 5:- If V is a finite dimensional vector space over F, then any two bases of V have the same number of elements.

Theorem 6:- If V(F) is a finite dimensional vector space then every Linearly independent subset of V is either a basis of V or can be extended to form a basis of V.

Theorem 7:- Let V be a finite-dimensional vector space and let dim V = n. Then

(i) any subsets of V which contains more than n vectors~~space and let dim V=n~~. is L.D.

(ii) no subset of V which contains less than n vectors can span V.