

Basis of a Vector Space: Let  $V$  be a vector space over a field  $F$  and let  $S$  be a non-empty subset of  $V$ , then  $S$  is said to be a basis of  $V$

- if
- $S$  is linearly independent (i.e.,  $\sum_{i=1}^n \alpha_i s_i = 0$  implies  $\alpha_i = 0$  for all  $i$ )
  - $L(S) = V$  i.e. every element of  $V$  is a linear combination of finite elements of  $S$ .

Ex.  $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  forms a basis of  $V(\mathbb{R}^3)$ , and is called usual basis.

$\{V(\mathbb{R}^3)\}$  is read as: Vector space  $V$  over real numbers ( $\mathbb{R}$ ) in 3-dimensions.

- The zero space has no basis.
- Every finitely generated vector space has a basis.
- Every non-zero vector space has a basis.
- A vector space may have more than one basis.

Finite Dimensional Vector Space:-

Let  $V(F)$  be a vector space over a field  $F$  and let  $S$  be a non-empty subset of  $V$ , then  $V(F)$  is said to be finite dimensional if  $S$  is finite subset of  $V$  such that  $L(S) = V$ .

If the set contains  $n$  elements, then the dimension of  $V$  is  $n$ .

## General Theorem

Theorem 1: If  $S = \{v_1, v_2, v_3, \dots, v_n\}$  is the basis of a vector space  $V(F)$ , then each element of  $V$  is uniquely expressible/represent as a linear combination of elements of  $S$ .

Proof: Since  $V = L(S)$ , each  $v \in V$  is expressible as  
Let two different sets  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  and  $(\beta_1, \beta_2, \dots, \beta_n)$  be scalars corresponding to an element  $S \in V$  such that

$$S = \alpha_1 S_1 + \alpha_2 S_2 + \dots + \alpha_n S_n \text{ and } S = \beta_1 S_1 + \beta_2 S_2 + \dots + \beta_n S_n \quad \text{--- (1)}$$

$$\alpha_1 S_1 + \alpha_2 S_2 + \dots + \alpha_n S_n = \beta_1 S_1 + \beta_2 S_2 + \dots + \beta_n S_n$$

$$\alpha_1 S_1 - \beta_1 S_1 + \alpha_2 S_2 - \beta_2 S_2 + \dots + \alpha_n S_n - \beta_n S_n = 0$$

$$(\alpha_1 - \beta_1) S_1 + (\alpha_2 - \beta_2) S_2 + \dots + (\alpha_n - \beta_n) S_n = 0$$

Since set  $S = (v_1, v_2, \dots, v_n)$  is linear independent so that

$$\alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, \dots, \alpha_n - \beta_n = 0$$

$$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n$$

Hence, the expression (1) is unique.

Theorem 2: - (Existence Theorem): Every finitely generated vector space has a finite basis.

Theorem 3 :- If  $V$  is a finite dimensional vector space and if  $S_1, S_2, S_3, \dots, S_m$  span  $V$ , then some subsets of  $S_1, S_2, \dots, S_m$  forms a basis of  $V$ .

Theorem 4 :- If  $(S_1, S_2, \dots, S_n)$  is a basis of  $V(F)$  and  $(V_1, V_2, V_3, \dots, V_m) \subset V$ , are linearly independent over  $F$ , then  $m \leq n$ .

Theorem 5 :- If  $V$  is a finite dimensional vector space over  $F$ , then any two bases of  $V$  have the same number of elements.

Theorem 6 :- If  $V(F)$  is a finite dimensional vector space then every linearly independent subset of  $V$  is either a basis of  $V$  or can be extended to form a basis of  $V$ .

Theorem 7 :- Let  $V$  be a finite-dimensional vector space and let  $\dim V = n$ . Then

- (i) any subsets of  $V$  which contains more than  $n$  vectors ~~and let  $\dim V = n$~~  is L.D.
- (ii) no subset of  $V$  which contains less than  $n$  vectors can span  $V$ .