Basis of a Vector space:

A subset S of a vector space V over a Field F i.e. V(F) is said to be a basis of V(F), if follows these properties.

(i) S consists of Linear independent vectors.

(ii) S generates V(F) i.e $L(S)=V$ i.e each vector in V is a Linear combination of a finite number of elements of S.

$V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_k v_k$

where $v_1, v_2, v_3, \cdots v_k$ are distinct vectors from S and $\alpha_1, \alpha_2, \cdots \alpha_k \in F$ (real numbers)

Example (i) Standard basis for $V(R^n)$ or $V_n(R)$

$e_1 = (1,0,0,\cdots 0,0)$, $e_2 = (0,1,0,0,\cdots 0,0)$

$e_3 = (0,0,1,0,0 \cdots 0,0) \cdots e_n = (0,0,0,\cdots 0,0,1)$

(ii) Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for $M_{2\times 2}(R)$

(iii) Polynomials $1, x, x^2, \cdots x^{n-1}$ form a basis for

$P_n = \{ a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1};\ a_i \in R \}$

(iv) The infinite set $\{1, x, x^2, \cdots x^n, \cdots\}$ is a basis for P, the space of all polynomials.

(v) The zero space $\{0\}$ has no basis

(vi) Every non-zero vector space has a basis

(vii) Every finite generated vector space has basis.

(viii) A vector space may have more than one basis.

Exp Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis of $V(R^3)$ and is called usual basis

$V(R^3)$ read as vectorspace V over R (real numbers) in 3-dimensions.

Exp. Determine whether or not the vectors $(1,-3,2)$, $(2,4,1)$ and $(1,1,1)$ form a basis of R^3?

Sol. We know dimension of $R^3 = 3$.

If we show that the given vectors are linearly independent, then they form a basis of R^3.

Let $\alpha, \beta, \gamma \in R$ i.e. 3 scalars.

then

$$\alpha(1,-3,2) + \beta(2,4,1) + \gamma(1,1,1) = 0$$

$$\begin{aligned} \alpha + 2\beta + \gamma &= 0 \quad \text{-(i)} \\ -3\alpha + 4\beta + \gamma &= 0 \quad \text{-(ii)} \\ 2\alpha + \beta + \gamma &= 0 \quad \text{-(iii)} \end{aligned} \Longrightarrow \begin{vmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix} = A$$

$$|A| = 1(4-1) - 2(-3-2) + 1(-3-8)$$

$$= 3 + 12 - 11 = 4 \neq 0$$ So the above Eqn possesses only trivial solution $\alpha = \beta = \gamma = 0$

Since the determinent of vectors (matrix A) is not equal to zero i.e $|A| \neq 0$, So the given vectors are Linearly independent and so So form of basis of R^3 is set of vectors.

$S = \{(1,-3,2), (2,4,1), (1,1,1)\}$ forms a basis of R^3.

Exp. Show that the set $S = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(4,5,0)}\}$ is not a basis of R^3

Sol. The set S of vectors written in matrix and find determinent.

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow |A| = 1(0) - 0(0) + 4(0) = 0$$

$$|A| = 0$$

Thus S is linear dependent and so S cannot be a basis of R^3

* **Dimension of a Vector Space** - If a vector space V over F has a finite basis having n vectors, then n is called the dimension of V over F and is written as $\dim_F(V) = n$ or $\dim V(F) = n$

→ If V does not have a finite basis, then V is called an infinite dimensional vector space.

Remark - (i) The dimension of null space $\{0\}$ is defined as zero

(ii) If $\dim V(F) = n$, then V has a basis containing n vectors, say $S = \{v_1, v_2, --- v_n\}$

The vectors $v_1, v_2, v_3, --- v_n$ are linearly independent over F and each $v \in V$ is expressible as $v = \alpha_1 v_1 + \alpha_2 v_2 + --- \alpha_n v_n$, $\alpha_i \in F$

(iii) If V is a finite dimension vector space (F-DVS) then any two basis of V have the same number of elements. Hence the dimension of V is uniquely defined.

Exps. (i) $\dim R^2 = 2$, since $\{(1,0),(0,1)\}$ is a basis of R^2

(ii) $\dim R^3 = 3$, since $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of R^3

(iii) $\dim R^n = n$, since $\{e_1 = (1,0,0,---0), e_2 = (0,1,0 --- 0)$ $------ e_n = (0,0,0, \cdots 1)\}$ is basis of R^n.

(iv) $\dim_R C$ or $\dim C(R) = 2$, since $\{1, i\}$ is basis of C over R

(v) If F is any field, then $\dim_F F = 1$; since $\{1\}$, consisting of the unity of F, is a basis of F over F.

In particular, $\dim_R R = 1$, $\dim_C C = 1$

(vi) Let $F_n[x]$ be the vector space of all polynomials (over a field F) with degree less than n. Then $\dim F_n[x] = n$, since $\{1, x, x^2, \cdots x^{n-1}\}$ is a basis of $F_n[x]$ over F.