GROUP THEORY

AUTOMORPHISM OF A GROUP :-

An isomorphism of a group onto itself is called an automorphism of a group.

OR

A one-one mapping f of a group G onto itself is called an automorphism if

$$f(ab) = f(a)\,f(b) \quad \forall\ a, b \in G.$$

Inner automorphism :- Let a be a fixed element of a group G. Then the map $f_a : G \to G$ such that $f_a(x) = a^{-1}xa\ \forall x \in G$ is an automorphism of G, known as inner automorphism.

Examples of Automorphism of a Group :-

(1) The permutation defined by $\begin{bmatrix} 1 & -1 & i & -i \\ 1 & -1 & -i & i \end{bmatrix}$ is an automorphism of the multiplication group of the set $\{1, -1, i, -i\}$ of four members.

(2) We consider the mapping f of the addition group C of all complex numbers into itself defined by

$$f : C \to C$$

such that $f(z) = \bar{z}\ \forall z \in G$

where $\bar{z}$ denotes the conjugate of the complex number z, is an automorphism of C.

THEOREMS ON AUTOMORPHISM :-

Theorem :- Let G be a group. Then a mapping which associates to each element $a \in G$, its inverse a^{-1} is an automorphism of G if and only if G is abelian.

Proof :- First assume that G is an abelian group.

Consider a mapping

$$f : G \to G$$

defined by $f(a) = a^{-1} \ \forall \ a \in G$

Then we have to prove that f is an automorphism.

To prove that f is one-one

Let a, b be any two elements of G such that

$$f(a) = f(b) \Rightarrow a^{-1} = b^{-1}$$ [By the definition of f]

$$\Rightarrow (a^{-1})^{-1} = (b^{-1})^{-1} \Rightarrow a = b$$

$\therefore f$ is one-one.

To prove that f is onto.

Let a be an arbitrary element of G, then $a^{-1} \in G$

$$\therefore f(a^{-1}) = (a^{-1})^{-1} = a .$$

Thus f is onto.

To prove that f is composition preserving let a, b be any two elements of G, then

$$f(ab) = ab^{-1} \quad (ab \in G)$$
$$= (ba)^{-1} \quad (\because G \text{ is abelion})$$
$$= a^{-1}b^{-1} \quad (\text{By reversal rule})$$
$$= f(a)\,f(b)$$

Hence f is an automorphism of G.

Conversely, we suppose a mapping f defined by

$$f : G \to G$$

Such that $f(a) = a^{-1}$ $\forall$ $a \in G$ is an automorphism.

Now we show that G is abelion.

Let a, b be any two elements of G, then

$$ab = (a^{-1})^{-1}(b^{-1})^{-1}$$
$$= f(a^{-1})\,f(b^{-1}) \quad (\text{By the definition of } f)$$
$$= f(a^{-1}b^{-1}) \quad (\because f \text{ preserves the composition})$$
$$= f((ba)^{-1}) \quad (\text{By reversal rule})$$
$$= ((ba)^{-1})^{-1} \quad (\text{By the definition of } f)$$
$$= ba$$

$\therefore$ $ab = ba$.

Hence G is abelian.