Attraction of a thin uniform rod

Newton's law of attraction

If m, m' denotes the masses of two particles and r their distance apart, then the force of attraction between them is $\gamma \frac{mm'}{r^2}$ where γ is known as the gravitational constant.

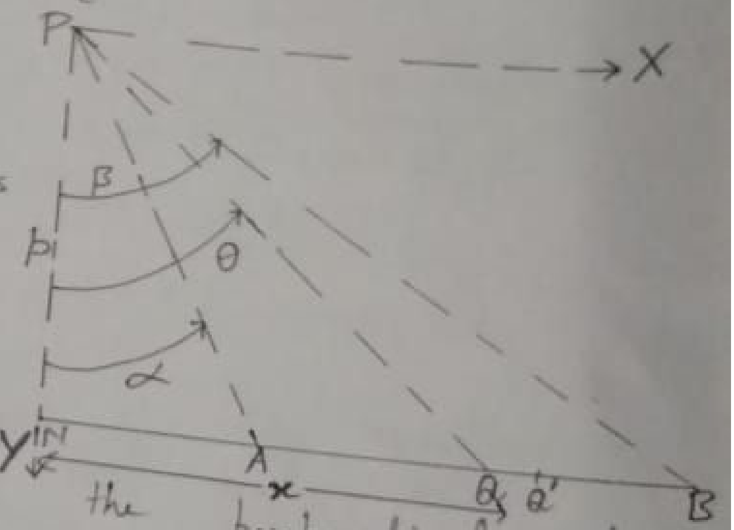
In CGS units, $\gamma = 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2}$

In astronomical units, $\gamma = 1$.

Q.) Find the attraction of a thin uniform rod at an external point.

Let AB be a thin uniform rod linear density m. [$m = k\rho$, k = cross sectional area. ρ = density.]

Let P be a point external to AB.



Let the perpendicular from P to AB meet AB in N. For simplicity we take N on BA produce though this is not necessary for the argument.

Let $NP = p$, $\angle NPA = \alpha$, $\angle NPB = \beta$

Take any point Q on the rod and suppose $NQ = x$ and $\angle NPQ = \theta$.

Then $x = p \tan\theta$

Let $QQ' = dx$ be an element of the rod at Q.

Then $QQ' = p \sec^2\theta \, d\theta$

$\therefore$ mass of the element $QQ' = m p \sec^2\theta\, d\theta$

$\therefore$ The attraction at P of the element QQ'

$$= \frac{\gamma m p \sec^2\theta\, d\theta}{PQ^2}$$

$$= \frac{\gamma m\, p \sec^2\theta\, d\theta}{(p\sec\theta)^2} = \frac{\gamma m}{p}\, d\theta$$

The direction of this attraction is ultimately along PQ when QQ' is very small.

The components of this attraction along and perpendicular to PN are respectively $\frac{\gamma m}{p}\cos\theta\, d\theta$ and $\frac{\gamma m}{p}\sin\theta\, d\theta$.

$\therefore$ The component of attraction at P of the whole rod AB along PN is

$$Y = \int_{\alpha}^{\beta} \frac{\gamma m}{p}\cos\theta\, d\theta$$

$$= \frac{\gamma m}{p}(\sin\beta - \sin\alpha) \text{ —— } (1)$$

and the component perpendicular to PN is

$$X = \int_{\alpha}^{\beta} \frac{\gamma m}{p}\sin\theta\, d\theta$$

$$= \frac{\gamma m}{p}(\cos\alpha - \cos\beta) \text{ —— } (2)$$

$\therefore$ If R be the magnitude of resultant attraction at P of the AB, we have

$$R^2 = X^2 + Y^2$$

$$= \left(\frac{\gamma m}{p}\right)^2 \left\{2 - 2\cos(\beta - \alpha)\right\}$$

$$= \left(\frac{\gamma m}{p}\right)^2 \cdot 4\sin^2\frac{\beta-\alpha}{2}$$

$$\therefore R = \frac{2\gamma m}{p}\sin\frac{\beta-\alpha}{2}$$

$$= \frac{2\gamma m}{p}\sin\frac{1}{2}APB \quad \text{—— ③}$$

Also if the direction of R makes an angle ϕ with PN, we have

$$\tan\phi = \frac{X}{Y} = \frac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha}$$

$$= \frac{2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}}{2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}}$$

$$= \tan\frac{\alpha+\beta}{2}$$

or, $$\phi = \frac{1}{2}(\alpha+\beta) \quad \text{—— ④}$$

Thus the direction of the resultant attraction bisects the angle APB.

③ and ④ determine the attraction at P completely.

[Note:— From ② we see that the attraction at P parallel to AB $\propto \frac{1}{PA} - \frac{1}{PB}$

Corollary :- Attraction of infinite rod.

When the rod is infinite in both sides, then $\alpha \to -\frac{\pi}{2}$ and $\beta \to \frac{\pi}{2}$

Hence from ①, $Y = \frac{2\gamma m}{p}$ and
from ②, $X = 0$

$$R = \frac{2\gamma m}{p} \quad \text{and} \quad \phi = 0$$

Thus the resultant attraction at P in this case $\frac{2\gamma m}{p}$ and its direction is perp. to the rod.

[clearly R varies inversely as the distance from the rod.]