Problems on Equipotential surfaces

Q (2) Show that the system of coaxial cyliders $x^2+y^2+2\lambda x+c^2=0$ can form a system of equipotential surfaces.

Ans. The given system is

$$x^2+y^2+2\lambda x+c^2=0 \quad \text{——— (1)}$$

where λ is a parameter and c an absolute constant.

Differentiating partially w.r.t. x we have

$$x+x\frac{\partial\lambda}{\partial x}+\lambda=0$$

$$\text{or, } \frac{\partial\lambda}{\partial x}=-1-\frac{\lambda}{x}$$

$$\therefore \frac{\partial^2\lambda}{\partial x^2}=\frac{\lambda}{x^2}-\frac{1}{x}\frac{\partial\lambda}{\partial x}$$

$$=\frac{\lambda}{x^2}+\frac{1}{x}\left(1+\frac{\lambda}{x}\right)$$

$$=\frac{2\lambda}{x^2}+\frac{1}{x}$$

similarly differentiating (1) partially w.r.t. y we get

$$y+x\frac{\partial\lambda}{\partial y}$$

$$\text{or, } \frac{\partial\lambda}{\partial y}=-\frac{y}{x}$$

$$\therefore \frac{\partial^2\lambda}{\partial y^2}=-\frac{1}{x}$$

$$\therefore \frac{\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}}{\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(\frac{\partial\lambda}{\partial y}\right)^2}=\frac{\frac{2\lambda}{x^2}+\frac{1}{x}-\frac{1}{x}}{\left(\frac{\lambda}{x}+1\right)^2+\frac{y^2}{x^2}}$$

$$= \frac{2\lambda}{\lambda^2 - c^2} = \text{a function of } \lambda \text{ only}$$

Hence ① is a possible family of equipotential surfaces in free space.

[Potential $V = f(\lambda)$ is given by $\frac{f''(\lambda)}{f'\lambda} = \frac{-2\lambda}{\lambda^2 - c^2}$

Integrating, $\log f'(\lambda) = \log A - \log(\lambda^2 - c^2)$

or, $f'(\lambda) = \frac{A}{\lambda^2 - c^2}$

$\therefore V = f(\lambda) = B + \frac{A}{2c} \log \frac{\lambda - c}{\lambda + c}$]