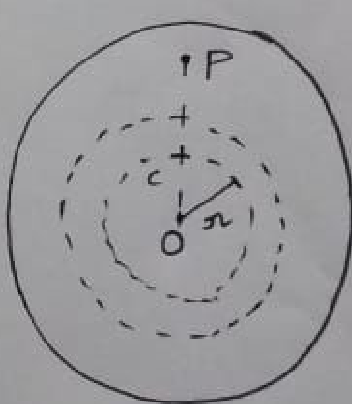
Q. (4) The density at a distance r from the centre, in a sphere of attracting matter is $\rho_0 - \lambda r$, where ρ_0 is the density at the centre and λ is a constant. Find the potential at any internal point and prove that the potential at the centre is

$$4\pi \gamma a^2 \left(\frac{\rho_0}{6} + \frac{\rho_1}{3} \right)$$

where a is the radius and ρ_1 the density at the surface of the sphere, and γ the constant of gravitation.

The mass of an elementary shell of radius r is



$$dM = 4\pi r^2 dr \cdot (\rho_0 - \lambda r)$$

Let P be any internal point in the sphere where $OP = c$, O being the centre of the sphere.

Then potential at P due to the elementary shell

$$= \begin{cases} \gamma \cdot \dfrac{4\pi r^2 dr\,(\rho_0 - \lambda r)}{c} & \text{if } r < c \\[2ex] \gamma \cdot \dfrac{4\pi r^2 dr\,(\rho_0 - \lambda r)}{r} & \text{if } r > c \end{cases}$$

$\therefore$ The whole potential at P due to the sphere is

$$V = \gamma \cdot \frac{4\pi}{c} \int_0^c r^2 (\rho_0 - \lambda r)\, dr + \gamma \cdot 4\pi \int r(\rho_0 - \lambda r)\, dr$$

$$= \gamma \cdot \frac{4\pi}{c} \left\{ \frac{\rho_0 c^3}{3} - \frac{\lambda c^4}{4} \right\} + \gamma \cdot 4\pi \left\{ \rho_0 \frac{a^2 - c^2}{2} - \lambda \frac{a^3 - c^3}{3} \right\}$$

$$= 4\pi\gamma \left[\rho_0 \left\{ \frac{c^3}{3} + \frac{a^2 - c^2}{2} \right\} - \lambda \left\{ \frac{c^3}{4} + \frac{a^3}{3} - \frac{c^3}{3} \right\} \right]$$

$$= 4\pi\gamma \left[\rho_0 \left\{ \frac{c^2}{3} + \frac{a^2 - c^2}{2} \right\} - \frac{\rho_0 - \rho_1}{a} \left\{ \frac{c^3}{4} + \frac{a^3}{3} - \frac{c^3}{3} \right\} \right]$$

$$(\because \rho_1 = \rho_0 - \lambda a)$$

$$= 4\pi\gamma \left[\rho_0 \left\{ \frac{c^2}{3} + \frac{a^2}{2} - \frac{c^2}{2} - \frac{c^3}{4a} - \frac{a^2}{3} + \frac{c^3}{3a} \right\} + \rho \left\{ \frac{c^3}{4a} + \frac{a^2}{3} - \frac{c^3}{3a} \right\} \right]$$

This gives the potential at $r = c$.

The potential at O (i.e. $c = 0$ is given by

$$V_0 = 4\pi\gamma \left[\rho_0 \left(\frac{a^2}{2} - \frac{a^2}{3} \right) + \rho_1 \frac{a^2}{3} \right]$$

$$= 4\pi\gamma a^2 \left(\frac{\rho_0}{6} + \frac{\rho_1}{3} \right)$$