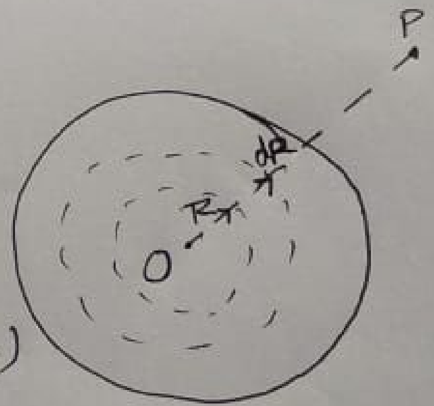
Q(2) A solid sphere of radius a is such that its density at any point is proportional to the n^{th} power of the distance of that point from the centre of the sphere. Find the potential at any external point, and show that the potential at an internal point at a distance r from the centre is $\frac{\gamma M}{n+2}\left(\frac{n+3}{a} - \frac{r^{n+2}}{a^{n+3}}\right)$ where M is the total mass of the sphere.



The mass of an elementary shell of radius R of the sphere is

$$dM = 4\pi R^2 dR \cdot \lambda R^n \ (\lambda = \text{Const.})$$

$\therefore$ Mass of the sphere is

$$M = 4\pi\lambda \int_0^a R^{n+2}\, dR$$

$$= 4\pi\lambda \frac{a^{n+3}}{n+3} \quad \text{———} \quad (1)$$

(i) Let P be any external point. Then the potential at P due to the elementary shell $= \gamma \cdot \frac{4\pi R^2 dR \cdot \lambda R^n}{OP}$

(where O is the centre of the shell)

Potential at P due to the whole sphere

$$= \gamma \int_0^a \frac{4\pi R^2 dR \lambda R^n}{OP}$$

$$= \frac{\gamma M}{OP} \text{ (from (1))}$$

(ii) Let P be an internal point with $OP = r$. Then potential at P due to an elementary shell of radius $R (< r)$

$$= \gamma \cdot \frac{4\pi R^2 dR \cdot \lambda R^n}{r}$$ (because P is external to such a shell)

Also the potential at P due to an elementary shell of radius $R (> r)$

$$= \gamma \cdot \frac{4\pi R^2 dR \cdot \lambda R^n}{R}$$ ($\because$ P is an internal point for such a shell)

$\therefore$ Potential at P due to the whole sphere is

$$V = \gamma \cdot \int_0^r \frac{4\pi R^2 dR \cdot \lambda R^n}{r} + \gamma \int_r^a \frac{4\pi R^2 dR \cdot \lambda R^n}{R}$$

$$= \frac{4\pi\lambda\gamma}{r} \frac{r^{n+3}}{n+3} + 4\pi\lambda\gamma \frac{a^{n+2} - r^{n+2}}{n+2}$$

$$= 4\pi\lambda\gamma \left\{ r^{n+2} \left(\frac{1}{n+3} - \frac{1}{n+2} \right) + \frac{a^{n+2}}{n+2} \right\}$$

$$= 4\pi\lambda r \left[\frac{-r^{n+2}}{(n+3)(n+2)} + \frac{a^{n+2}}{n+2} \right]$$

$$= r \cdot \frac{4\pi\lambda a^{n+3}}{n+3} \left\{ \frac{n+3}{a(n+2)} - \frac{r^{n+2}}{a^{n+3}} \cdot \frac{1}{n+2} \right\}$$

$$= r M \cdot \frac{1}{n+2} \left\{ \frac{n+3}{a} - \frac{r^{n+2}}{a^{n+3}} \right\} \quad (\text{from } 1)$$