Poisson's equation (or theorem) on potential

Statement :- If V is the potential of a body at a point in contact with the attracting matter at which the density ρ is finite and continuous then

$$\nabla^2 V = -4\pi\gamma\rho$$

Proof :- We know that the potential V and the intensity (of attraction) $\vec{F}$ are finite and continuous throughout the space. Where the density ρ is finite, and are connected by the equation.

$$\vec{F} = \nabla V \quad \text{——— (1)}$$

Moreover, the derivatives of the intensity are also finite and continuous where ρ is finite and continuous.

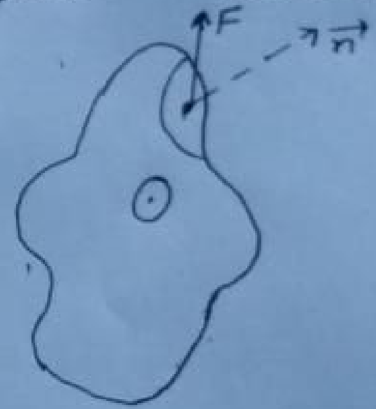
Therefore, if S is any closed surface (enclosing a region V) drawn within the space in which ρ is finite and continuous by Gauss theorem on total normal intensity (which states that the total normal intensity of the force of attraction over any closed surface S in a gravitational field of force is equal to $(-4\pi\gamma)$ times the mass enclosed by S) we have

$$-4\pi\gamma\int_V \rho \, dv = \int_S \vec{F}\cdot\vec{n}\, ds \quad \text{—— (2)}$$

where $\vec{n}$ is the outward unit normal at any point P on S and ds is the surface element surrounding P.

But, by Gauss divergence theorem

$$\int_S \vec{F}\cdot\vec{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$



$\therefore$ From (2), we have

$$-4\pi\gamma\int_V \rho\, dv = \int_V \operatorname{div}\vec{F}\, dv$$

$$= \int_V \operatorname{div}(\nabla V)\, dv, \text{ by (1)}$$

$$= \int_V \nabla^2 V\, dv$$

$$\therefore \int_V (\nabla^2 V + 4\pi\gamma\rho)\, dv = 0 \quad \text{—— (3)}$$

since v is arbitrary, (3) implies that

$$\nabla^2 V + 4\pi\gamma\rho = 0$$

$$\text{or, } \nabla^2 V = -4\pi\gamma\rho$$

Q 3 (i) Define an equipotential family of surfaces.

(ii) show that the attraction at any point of an equipotential surface is normal to the surface.

Ans. (i) A family of surfaces

$$\phi(x, y, z) = c \quad \text{--- (1)}$$

where c is a parameter) is called a family (or system) of equipotential surfaces if the potential V at any point (x, y, z) of (1) depends solely on c.

(ii) Let $\vec{F}$ be the force of attraction at any point $\vec{r} = (x, y, z)$ of an equipotential surface $\phi(x, y, z) = c$ --- (1)

Due to displacement $\vec{dr}$ along the surface (1), the work done by $\vec{F}$ is $\vec{F} \cdot \vec{dr}$. Since the difference of potentials on any two points on the equipotential (1) is zero, it follows that no work is done by $\vec{F}$ due to any displacement $\vec{dr}$ along the surface.

Hence,

$$\vec{F} \cdot \vec{dr} = 0$$

$\therefore \vec{F}$ is normal to the surface (1).