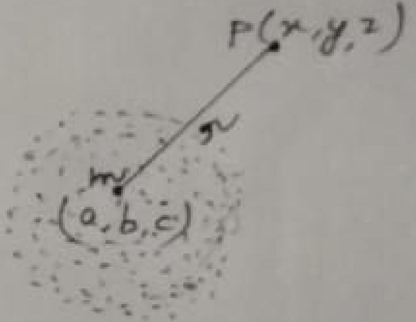
Laplace's theorem on potential

Statement:- If V be the potential of an attracting system at any point which does not concide with any of the attracting particles then

$$\nabla^2 V = 0$$

Proof:-

First suppose that the attracting system consists of a discrete distribution of particles



Let $P(x, y, z)$ be any point unoccupied by particles of the system, i.e. let P be a point in free space.

Let m be the mass of a particle at the point (a, b, c) situated at a distance r from P. Then the potential at P of the attracting system is given by

$$V = \sum \frac{\gamma m}{r} \quad \text{———} \quad (1)$$

where $\sum$ denotes summation over all the particles of the system and

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad \text{———} \quad (2)$$

Differentiating 2 partially w.r.t. x we have

$$2r \frac{\partial r}{\partial x} = 2(x-a)$$

so that $\dfrac{\partial r}{\partial x} = \dfrac{x-a}{r}$

similarly $\dfrac{\partial r}{\partial y} = \dfrac{y-b}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z-c}{r}$

Now, differentiating 1 partially w.r.t. x we get

$$\frac{\partial V}{\partial x} = -\sum \frac{\gamma m}{r^2} \cdot \frac{\partial r}{\partial x} = -\gamma \sum \frac{m(x-a)}{r^3}$$

$$\therefore \frac{\partial^2 V}{\partial x^2} = -\gamma \sum m \left\{ \frac{1}{r^3} - \frac{3(x-a)}{r^4} \frac{\partial r}{\partial x} \right\}$$

$$= -\gamma \sum m \left\{ \frac{1}{r^3} - \frac{3(x-a)^2}{r^5} \right\}$$

similarly $\dfrac{\partial^2 V}{\partial y^2} = -\gamma \sum m \left\{ \dfrac{1}{r^3} - \dfrac{3(y-b)^2}{r^5} \right\}$

and $\dfrac{\partial^2 V}{\partial z^2} = -\gamma \sum m \left\{ \dfrac{1}{r^3} - \dfrac{3(z-c)}{r^5} \right\}$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

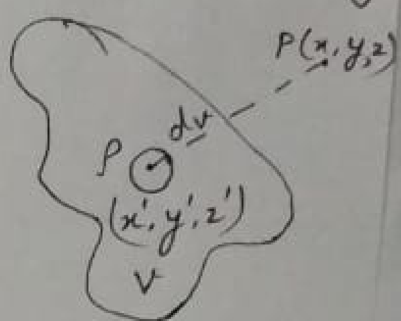
$$= -\gamma \sum m \left[\frac{3}{r^3} - \frac{3}{r^5} \left\{ (x-a)^2 + (y-b)^2 + (z-c)^2 \right\} \right]$$

$$= -\gamma \sum m \left[\frac{3}{r^3} - \frac{3}{r^5} \cdot r^2 \right] \text{ by } 2$$

This proves the theorem for a discrete distribution of matter.

Next suppose that the attracting system consists of a continuous body and $P(x, y, z)$ is any point not in contact with any particle of the body.

Let V be the volume of the body, dv an elementary volume of the body at a point (x', y', z') where density of matter is ρ. Let r be the distance of (x', y', z') from given point P.



Then r is never zero for any point of the body. Also

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \text{ so that}$$

$$\frac{\partial r}{\partial x} = \frac{x-x'}{r} \quad \text{etc.}$$

Now the potential at P of the body is given by

$$V = \iiint \frac{\rho \, dv}{r}$$

since $r \neq 0$

$\therefore$ The above integral is convergent and differentiation under the sign of integration is permissible.

$$\therefore \frac{\partial V}{\partial x} = -\gamma \int_V \frac{\rho\, dv}{r^2} \frac{\partial r}{\partial x}$$

$$= -\gamma \int_V \frac{\rho\, dv}{r^2} \cdot \frac{x - x'}{r}$$

$$= -\gamma \int_V \frac{\rho\, dv}{r^3} (x - x')$$

$$\therefore \frac{\partial^2 V}{\partial x^2} = -\gamma \int_V \left\{ \frac{\rho\, dv}{r^3} - \frac{3\rho\, dv}{r^4} \frac{\partial r}{\partial x} \cdot (x - x') \right\}$$

$$= -\gamma \int_V \left\{ \frac{\rho}{r^3} - \frac{3\rho}{r^5} (x - x')^2 \right\} dv$$

similarly $\frac{\partial^2 V}{\partial y^2} = -\gamma \int_V \left\{ \frac{\rho}{r^3} - \frac{3\rho}{r^5} (y - y') \right\} dv$

and $\frac{\partial^2 V}{\partial z^2} = -\gamma \int_V \left\{ \frac{\rho}{r^3} - \frac{3\rho}{r^5} (z - z')^2 \right\} dv$

Adding, we get

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= -\gamma \int_V \frac{3}{r^3} - \frac{3}{r^5} \left\{ (x - x')^2 + (y - y')^2 + (z - z')^2 \right\} \rho\, dv$$

$$= -\gamma \int_V \left[\frac{3}{r^3} - \frac{3}{r^3} \right] \rho\, dv = 0$$

This proves the theorem in case of a continuous distribution of matter.