Q.) Prove that if the surface density of a circular disc at any point distant r from the centre is λr then the attraction at a point on the axis at a distance h is

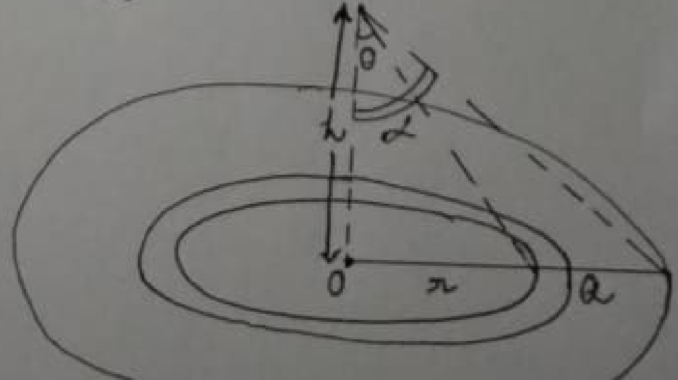
$$\pi \gamma \lambda h \left\{ \log \frac{\sqrt{a^2+h^2}+a}{\sqrt{a^2+h^2}-a} - \frac{2a}{\sqrt{a^2+h^2}} \right\}$$

where a = radius of disc.

With reference to the adjoining figure, the attraction at P (a point on the axis of the circle at a distance h from the centre O) due to an elementary ring of radius r and breadth dr is along PO and its magnitude is given by

$$dF = \frac{\gamma \cdot 2\pi r \, dr \, \lambda r}{PQ^2} \cos\theta$$

∴ Attraction at P due to the whole circular disc is along PO and its magnitude is



$$F = 2\pi\gamma\lambda \int_0^a \frac{r^2\, dr}{h^2+r^2} \frac{h}{\sqrt{h^2+r^2}}$$

$$= 2\pi\gamma\lambda h \int_0^a \frac{(r^2+h^2-h^2)\,dr}{(r^2+h^2)^{3/2}}$$

$$= 2\pi r \lambda h \int_0^a \frac{dx}{\sqrt{x^2+h^2}} - 2\pi r \lambda h^3 \int_0^a \frac{dx}{(x^2+h^2)^{3/2}}$$

$$= 2\pi r \lambda h \left\{ \left[\log x + \sqrt{h^2+x^2} \right]_0^a - 2\pi r \lambda h^3 \cdot I \right. \qquad \text{—— (1)}$$

Where $$I = \int_0^a \frac{dx}{(x^2+h^2)^{3/2}}$$

$$= \int_0^{\tan^{-1}\frac{a}{h}} \frac{h \sec^2\theta \, d\theta}{h^3 \cdot \sec^3\theta}$$

$$(\because x = h\tan\theta)$$

$$= \frac{1}{h^2} \left[\sin\theta \right]_0^{\tan^{-1}\frac{a}{h}}$$

$$= \frac{1}{h^2} \sin\left(\sin^{-1} \frac{a}{\sqrt{a^2+h^2}} \right)$$

$$= \frac{1}{h^2} \cdot \frac{a}{\sqrt{a^2+h^2}}$$

$\therefore$ From (1)

$$F = 2\pi r \lambda h \cdot \log \frac{a+\sqrt{a^2+h^2}}{h} - 2\pi r \lambda h \cdot \frac{a}{\sqrt{a^2+h^2}}$$

$$= \pi r \lambda h \left[\log \frac{(a+\sqrt{a^2+h^2})^2}{h^2} - \frac{2a}{\sqrt{a^2+h^2}} \right]$$

$$= \pi r \lambda h \left[\log \frac{(\sqrt{a^2+h^2}+a)(\sqrt{a^2+h^2}+a)(\sqrt{a^2+h^2}-a)}{h^2(a^2+h^2-a)} - \frac{2a}{a^2+h^2} \right]$$

$$= \pi r \lambda h \left[\log \frac{\sqrt{a^2+h^2}+a}{\sqrt{a^2+h^2}-a} - \frac{2a}{\sqrt{a^2+h^2}} \right]$$