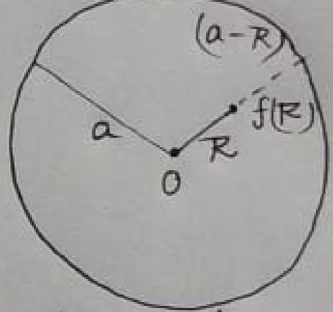
Q.) The density of a sphere varies as the depth below the surface. Show that the resultant attraction is greatest at a depth equal to $\frac{1}{3}$ of the radius and that the value is $\frac{4}{3}$ of the value at the centre.

Let a be the radius of the sphere. By question the density at a distance R from the centre O is $\rho = \lambda(a-R)$ where λ is a constant.



Now the attraction of a sphere at a distance r from the centre is given by

$$F = \frac{4\pi\gamma}{r^2}\int_0^r \rho \cdot R^2 \, dR$$

$$= \frac{4\pi\gamma\lambda}{r^2}\int_0^r (a-R)R^2 \, dR$$

$$= \frac{4\pi\gamma\lambda}{r^2}\left(\frac{ar^3}{3} - \frac{r^4}{4}\right)$$

$$= 4\pi\gamma\lambda\left(\frac{ar}{3} - \frac{r^2}{4}\right)$$

$$= \pi\gamma\lambda\left\{\frac{4}{9}a^2 - \left(\frac{2a}{3} - r\right)^2\right\}$$

Hence F is maximum when $r = \frac{2a}{3}$, i.e. F is maximum at a depth equal to $\frac{1}{3}a$ below the surface.

Moreover, the maximum value of F is given by

$$F_{max} = \frac{4\pi \sqrt{\lambda} a^2}{9} \quad \text{—— } (1)$$

But the value of F at the surface of the sphere is given by

$$F_s = \pi \sqrt{\lambda} \left\{ \frac{4a^2}{9} - \left(\frac{a}{3}\right)^2 \right\}$$

$$= \frac{1}{3} \pi \lambda a^2 \quad \text{—— } (2)$$

From (1) and (2)

$$F_{max} = \frac{4}{3} F_s$$, as required.