Potential

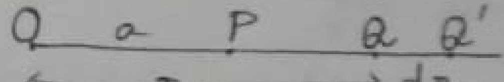
Definition :-

Let there be a mass m at the point O. Then the potential V at any point P if the mass m is defined to be the work that would be done by the attraction force of m on a particle of unit mass at its moves along any path from an infinite distance upto the point P.

Taking the path to be a straight line, the above definition gives

$$V = \gamma \int_{\infty}^{a} \left(-\frac{m}{r^2}\right) dr$$

$$= \gamma \left[\frac{m}{r}\right]_{\infty}^{a}$$



$$= \gamma \frac{m}{a}$$

$$= \frac{m}{a} \text{ in astronomical units } (\gamma = 1)$$

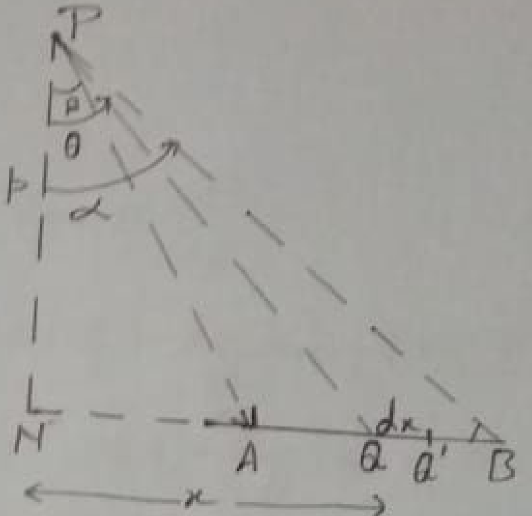
The potential V of m situated at O at a point P distant r from O is given by

$$V = \frac{m}{r}$$

$$\therefore \frac{dV}{dr} = \frac{-m}{r^2} = \text{force of attraction}$$

of m towards OP.

(1) (a) Find the potential of a rod at an external point.



Let the given rod be AB. Let k be the cross sectional area and ρ the density of the rod.

Let P be a point external to AB.

Let the perpendicular from P on AB meet AB in N. For simplicity we take N on BA produced though this is not necessary for the following argument.

Let $NP = p$, $\measuredangle NPA = \beta$, $\measuredangle NPB = \alpha$. Take any point Q on AB and suppose $NQ = x$, $\measuredangle NPQ = \theta$.

Then $x = p \tan\theta$.

Let $QQ' = dx$ be an element of the rod at Q, then

$$QQ' = p \sec^2\theta \, d\theta$$

and the mass of the element

$$QQ' = k\rho p \sec^2\theta \, d\theta$$

$\therefore$ The potential at P of the element $QQ' = \dfrac{\gamma m p \sec^2\theta \, d\theta}{PQ}$

$= \gamma m p \sec^2\theta \, d\theta$

$\therefore$ The potential at P of the rod AB is given by

$$V = \int_{\beta}^{\alpha} \gamma m \sec\theta \, d\theta$$

$$= \gamma m \left[\log \tan \tfrac{1}{2}\left(\theta + \tfrac{\pi}{2}\right)\right]_{\beta}^{\alpha}$$

$$= \gamma m \left[\log \tan \tfrac{1}{2}\left(\alpha + \tfrac{\pi}{2}\right) - \log \tan \tfrac{1}{2}\left(\beta + \tfrac{\pi}{2}\right)\right]$$

$$= \gamma m \left[\log \tan \tfrac{1}{2}\left(\pi - PBA\right) - \log \tan \tfrac{1}{2} PAB\right]$$

$$= \gamma m \left[\log \cot \tfrac{1}{2} PBA - \log \tan \tfrac{1}{2} PAB\right]$$

$$= \gamma m \log \left(\cot \tfrac{1}{2} PBA \cdot \cot \tfrac{1}{2} PAB\right)$$

Cor 1 :-

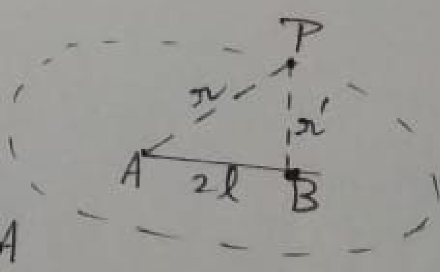
The potential of a rod AB is the same at every point of an ellipse with A and B as foci.

Proof :-

Let $PA = r$, $PB = r'$

$AB = 2l$

Put $r + r' + 2l = 2s$



Then $\cot \tfrac{1}{2} PAB \cdot \cot \tfrac{1}{2} PBA$

$$= \sqrt{\left\{\frac{s(s-r')}{(s-r)(s-2l)} \cdot \frac{s(s-r)}{(s-r')(s-2l)}\right\}}$$

$$= \frac{s}{s-2l}$$

$$= \frac{r + r' + 2l}{r + r' - 2l}$$

Now if A, B are the foci of the ellipse passing through P and $2a$ be the major axis of the ellipse, then

$$r + r' = 2a.$$

Hence the potential V at P

$$= \gamma m \log \frac{a+l}{a-l}$$

which is constant and independent of the position of P on the ellipse.

Hence the result.