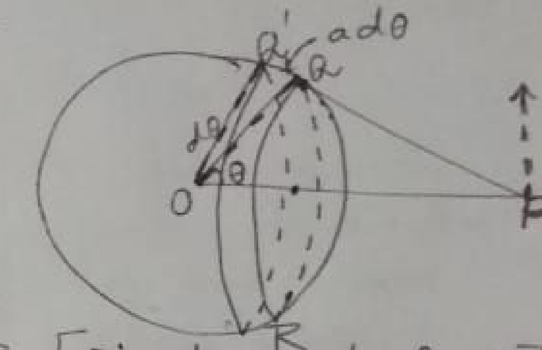
Q.) Find the attraction of a thin spherical shell at
(i) an external point
(ii) an internal point
(iii) a point on its surface

Let a be the radius, O the centre, k the thickness and ρ the density of the spherical shell.



Let P be any given point, join OP. [Fig for external P]
Let OP = C, take any point Q on the surface of the shell. Through Q draw a perpendicular to OP intersecting the shell in a circle with centre M on OP and radius $= MQ = a\sin\theta$ where $\angle QOP = \theta$
Draw a contiguous parallel plane ($\perp$ OP) intersecting the shell in a contiguous circle through a point Q' adjacent to Q. In this way we get a thin zone of the shell bounded by these two circles.

The breadth of this zone is $QQ' = a\,d\theta$ (where $Q'OP = d\theta$)

Area of this zone of the shell
$= 2\pi \cdot a\sin\theta\, a$

$\therefore$ mass of the zone $= 2\pi k\rho \cdot a^2 \sin\theta\, d\theta$

Each point of this zone is at the same distance from P.

Now the attraction at P of a unit mass placed at $Q = \frac{r}{PQ^2}$ along PQ. Its components along and perp. to PO are respectively $\frac{r}{PQ^2} \cdot \cos\phi$ and $\frac{r}{PQ^2} \sin\phi$ where $\phi = \angle OPQ$.

If R be the point on the zone diametrically opposite to Q, then attraction at P of a unit mass placed at $R = \frac{r}{PR^2}$ along PR.

Its components along PO is

$$\frac{r}{PR^2} \cos\phi = \frac{r}{PQ^2} \cos\phi$$

which is same as the component of attraction along PO due to the unit mass at Q. Also the component of attraction perp. to PO due to the unit mass at R is $\frac{r}{PR^2} \sin\phi$ i.e. $\frac{r}{PQ^2} \sin\phi$, which is equal and opposite to the component of attraction perp. to PO due to the unit mass at Q.

Hence the resultant attraction

③

at P due to the pair of unit masses at Q and R is $\frac{2\gamma}{PQ^2}\cos\phi$ and it acts along PO.

Hence the attraction at P due to the whole zone of shell is along PO and its magnitude

$$= \frac{1}{2}\left\{\frac{2\gamma}{PQ^2}\cos\phi \cdot 2\pi k\rho a^2 \sin\theta\, d\theta\right\}$$

But from $\Delta\, OQP$, $\cos\phi = (r^2+c^2-a^2)/2rc$,

$$r^2 = OQ^2 + OP^2 - 2OQ\cdot OP\cos\theta.$$

and so $2r\,dr = +2ac\sin\theta\, d\theta$

$\therefore$ The attraction at P due to the zone of the shell is along PO and its magnitude

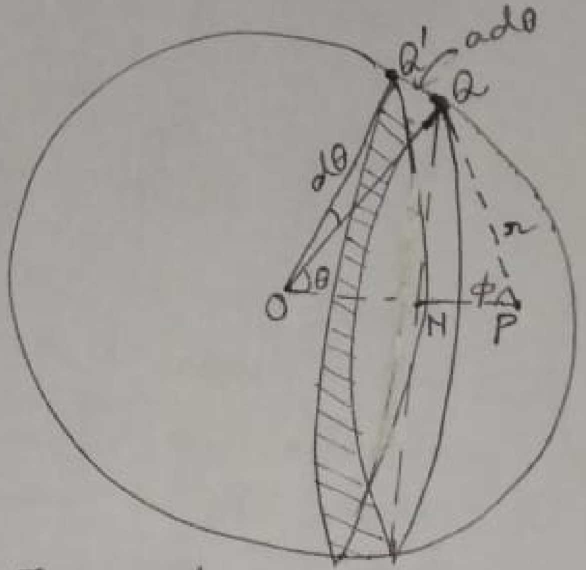
$$= \frac{\gamma}{r^2}\cdot\frac{r^2+c^2-a^2}{2rc}\cdot 2\pi a k\rho\cdot\frac{2r\,dr}{2c}$$

$$= \pi\gamma k\rho\frac{a}{c^2}\left(\frac{r^2+c^2-a^2}{r^2}\right)dr$$

$$= \frac{\pi\gamma k\rho a}{c^2}\left\{1+\frac{c^2-a^2}{r^2}\right\}dr \qquad \text{(A)}$$

(ii) suppose P is a point inside the shell. Then r varies from $a-c$ to $a+c$.

$\therefore$ From (A), the attraction at P due to the whole shell is along PO and its magnitude.



[Fig for internal P]

$$= \frac{\pi \gamma k \rho a}{c^2} \int_{a-c}^{a+c} \left\{ 1 + \frac{c^2 - a^2}{r^2} \right\} dr$$

$$= \frac{\pi \gamma k \rho a}{c^2} \left[r - \frac{c^2 - a^2}{r} \right]_{a-c}^{a+c}$$

$$= \frac{\pi \gamma k \rho a}{c^2} \left\{ 2c - \frac{c^2 - a^2}{c+a} + \frac{c^2 - a^2}{a-c} \right\}$$

$$= \frac{\pi \gamma k \rho a}{c^2} \left\{ 2c - (c-a) - (c+a) \right\}$$

$$= 0$$

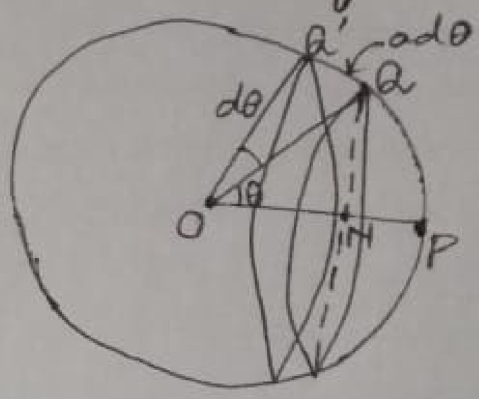
Hence the attraction of a shell at any internal point is zero.

(iii)

Suppose P is a point on the surface of the shell.

Then r varies from 0 to $2a$. Also $c = a$

$\therefore$ From (A) the attraction at P due to the whole shell is along PO and its magnitude is



$[NQ = a\sin\theta]$

[Fig. for P on the surface of the shell]

$$= \pi\gamma k\rho \frac{a}{c^2}\int_0^{2a}\{1-0\}\,dr$$

$$= \frac{2\pi a^2 k\rho\gamma}{a^2}$$

$$= \frac{1}{2}\cdot\frac{\gamma M}{a^2}$$