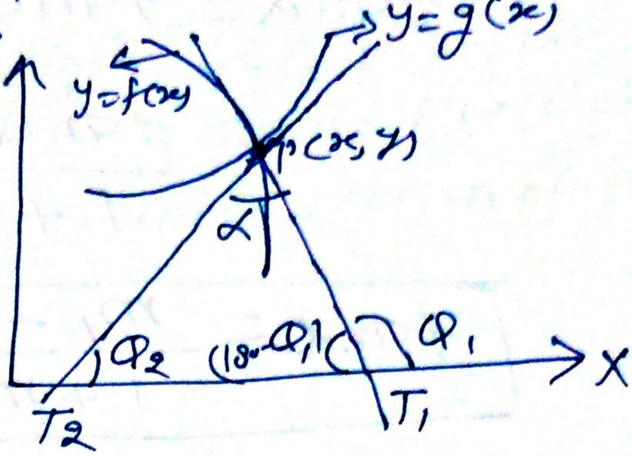
Angle of Intersection of two curves

Let the equations of both the curves be

$y = f(x)$

$y = g(x)$



Let the two curves cut each other at the point $P(x,y)$

From P, draw the tangent PT_1 to the curve $y=f(x)$ and the tangent PT_2 to the curve $y=g(x)$ which meet the x axis respectively at the points T_1 and T_2.

The angle between the two curves is α at P.

their tangents PT_1 and PT_2.

To find angle $\angle T_2PT_1 = \alpha$.

Let the tangent PT_1 make an angle θ_1 with the x-axis and PT_2 make an angle θ_2 with the x axis

i.e. $\angle PT_1X = \theta_1$

$\angle PT_2X = \theta_2$

Then $\angle T_2PT_1 = \theta_1 - \theta_2 = \alpha$

Since PT_1 make an angle θ_1 with x axis

$$\tan\theta_1 = \frac{dy}{dx} = \frac{d}{dx}(y=f(x)) = f'(x) = m_1$$

similarly, PT_2 make an angle θ_2 with x axis

$$\tan\theta_2 = \frac{dy}{dx} = \frac{d}{dx}(y=g(x)) = g'(x) = m_2$$

$\angle T_2 P T_1 = \alpha = \theta_1 - \theta_2$

$\alpha = \theta_1 - \theta_2$

$\tan\alpha = \tan(\theta_1 - \theta_2)$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 . \tan\theta_2}$$

$$\boxed{\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

This formula gives the angle between both the curves.

(i) If the two curves touch each other then $\alpha = 0$

$\therefore m_1 - m_2$

$\Rightarrow f'(x) - g'(x) = 0$

i.e $f'(x) = g'(x)$

(ii) If two curves cut each other at right angle, then $\alpha = 90^\circ$

$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ}$

$\therefore 1 + m_1 m_2 = 0$

$\frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1}{0}$

i.e. $1 + f'(x) g'(x) = 0$

$\Rightarrow 1 + \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \times \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$ putting the values of $f'(x)$ and $g'(x)$

$$\frac{\partial f}{\partial x} . \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \cancel{\times} \frac{\partial g}{\partial y} = 0$$

Exp. Find the angle of intersection of the curves.

$$x^2 - y^2 = a^2$$
$$x^2 + y^2 = \sqrt{2}a^2$$

Sol. We know that the angle between the curves is the same as the angle b/w their tangents at the point of intersection of the two curves.

The given curves

$$x^2 - y^2 = a^2 \quad \text{--- (I)}$$
$$x^2 + y^2 = \sqrt{2}a^2 \quad \text{--- (II)}$$

Adding Eqn (I) + (II)

$$2x^2 = (\sqrt{2}+1)a^2$$
$$x^2 = \frac{\sqrt{2}+1}{2}a^2$$
$$x = a\left(\frac{\sqrt{2}+1}{2}\right)^{1/2}$$

Subtracting Eqn (I) ~~from~~ (II)

$$2y^2 = (\sqrt{2}-1)a^2$$
$$y^2 = \left(\frac{\sqrt{2}-1}{2}\right)a^2$$
$$y = a\left(\frac{\sqrt{2}-1}{2}\right)^{1/2}$$

Thus the co-ordinates of their point of intersection are found $x = a\left(\frac{\sqrt{2}+1}{2}\right)^{1/2}$

$$y = a\left(\frac{\sqrt{2}-1}{2}\right)^{1/2}$$

Now differentiating curves w.r.t. x

$$x^2 - y^2 = a^2$$

$$2x - 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y} = m_1 = \tan\theta_1$$

and

$$x^2 + y^2 = \sqrt{2}\, a^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} = m_2 = \tan\theta_2$$

Let the angle b/w tangents be α that $\alpha = \theta_1 - \theta_2$

$$\tan\alpha = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan\alpha = \frac{\frac{x}{y} - \left(-\frac{x}{y}\right)}{1 + \left(\frac{x}{y}\right)\left(-\frac{x}{y}\right)} = \frac{\frac{2x}{y}}{1 - \left(\frac{x}{y}\right)^2}$$

$$\tan\alpha = \frac{2x/y}{\frac{x^2 - y^2}{y^2}} = \frac{2xy}{-x^2 + y^2} \quad \text{— (III)}$$

Putting x & y values from co-ordinate of their intersection point in Eqn (III)

$$\tan\alpha = \frac{\cancel{2}\frac{a}{\cancel{\sqrt{2}}}(\sqrt{2}+1)^{1/2} \cdot \frac{a}{\cancel{\sqrt{2}}}(\sqrt{2}-1)^{1/2}}{\frac{a^2}{2}(\sqrt{2}+1) - \frac{a^2}{2}(\sqrt{2}+1)}$$

$$\tan\alpha = \frac{\cancel{a^2}((\sqrt{2})^2 - 1)^{1/2}}{\frac{\cancel{a^2}}{2}(\cancel{\sqrt{2}}+1 - \cancel{\sqrt{2}}+1)} = \frac{(2-1)^{1/2}}{\frac{-\cancel{2}}{\cancel{2}}} = -1$$

$$\tan\alpha = \tan 45 \Rightarrow \alpha = 135^\circ \quad \text{or } \cancel{135^\circ}$$

$$\boxed{\because \tan 135^\circ = -1}$$