Analytical Geometry of two dimensions

Problems (PARABOLA) :-

Q.1 Find the Condition that the line $y = mx + c$ may touch the parabola $y^2 = 4ax$

Sol :- The equation of the parabola is

$$y^2 = 4ax \quad \text{--- (i)}$$

and the equation of the line is $y = mx + c$ --- (ii)

Eliminating y between (i) and (ii), we get

$$(mx + c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mxc + c^2 = 4ax$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$$

This is a quadratic equation if x. So it will have two roots in order that the line (ii) will become tangent to (i), the root of the above equation must be equal. The Condition for this is $D = 0$.

$$\text{i.e.} \quad (2mc - 4a)^2 - 4m^2c^2 = 0$$

$$\Rightarrow 4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 = 0$$

$$\Rightarrow 16a(a - mc) = 0$$

$$\Rightarrow a - mc = 0$$

$$\Rightarrow mc = a$$

$$\Rightarrow c = a/m$$

$\therefore$ $y = mx + \frac{a}{m}$ this is the required equation.

Q.) Find the Condition that the line $y = mx + c$ may touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Sol:- The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (i)}$$

and the equation of line is

$$y = mx + c \quad \text{--- (ii)}$$

Putting from (ii) in (i) we get

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{m^2x^2 + 2mxc + c^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2mxc + a^2c^2 - a^2b^2 = 0$$

This is the quadratic equation in x. So it will have two roots. If the line (ii) touches the ellipse (i), then the two values of x will be equal.

$$D = 0$$

$$\Rightarrow 4a^4m^2c^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$$

$$\Rightarrow a^4m^2c^2 - a^2b^2c^2 + a^2a^4 - a^4m^2c^2 + a^4b^2m^2 = 0$$