Analytical geometry of three dimensions

Q:) Find the angle between two straight lines whose direction Cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2)

Ans:-

Let OP_1 and OP_2 be the two lines through O parallel to the given lines whose direction Cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Let the Co-ordinates of P_1 and P_2 be respectively (x_1, y_1, z_1) and (x_2, y_2, z_2) and θ be the angle between the given lines.

Let $OP_1 = r_1$ and $OP_2 = r_2$

Then $r_1^2 = x_1^2 + y_1^2 + z_1^2$

and $r_2^2 = x_2^2 + y_2^2 + z_2^2$

Also $x_1 = l_1 r_1$, $y_1 = m_1 r_1$, $z_1 = n_1 r_1$

and $x_2 = l_2 r_2$, $y_2 = m_2 r_2$, $z_2 = n_2 r_2$

$$P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$= (x_2^2 + y_2^2 + z_2^2) + (x_1^2 + y_1^2 + z_1^2) - 2(x_1x_2 + y_1y_2 + z_1z_2)$$

$$= r_2^2 + r_1^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)$$

From ΔP_1OP_2, $\cos\theta = \dfrac{OP_1^2 + OP_2^2 - P_1P_2^2}{2OP_1 \cdot OP_2}$

$$= \frac{r_1^2 + r_2^2 - r_2^2 - r_1^2 + 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$$

$$= \frac{2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$$

Hence $\boxed{\cos\theta = l_1l_2 + m_1m_2 + n_1n_2}$

i.e. $\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$

COLLARIES :- From Lagrange's Identity

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$= (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$$

Now $\sin^2\theta = 1 - \cos^2\theta$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$= (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$$

Hence $\boxed{\sin\theta = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}}$

Q.) Prove that the straight lines whose direction cosines are given by $2l + 2m - n = 0$, $mn + nl + lm = 0$ are at right angles.

sol:- Given that $2l + 2m - n = 0$ —— ①

i.e. $n = 2l + 2m$ and $mn + nl + lm = 0$

i.e. $m \cdot (2l + 2m) + (2l + 2m)l + lm = 0$

i.e. $2l^2 + 5lm + 2m^2 = 0$

i.e. $(2l + m)(l + 2m) = 0$

If $2l + m = 0$, we have $n = m$, from ①

$$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{2}$$

i.e. the direction cosines one line are proportional to $-1, 2, 2$.

$\therefore$ Actual direction Cosines are $\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}$

If $l+2m=0$, then from (1) we have $n=l$;

$$\therefore \frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

i.e. the direction Cosines of the 2nd line are proportional to $2, -1, 2$.

Then the angle between the two lines $= \cos^{-1}[l_1 l_2 + m_1 m_2 + n_1 n_2]$.

Now eliminate n between the two equations.

$\therefore l^2 + m^2 - (l+m)^2 = 0$; or $2lm = 0$

$\therefore$ either $l = 0$ or, $m = 0$

If $l = 0$, $m + n = 0$; $\therefore m = -n$

But $l^2 + m^2 + n^2 = 1$; $\therefore 2m^2 = 1$

$$\therefore m^2 = \frac{1}{2}, \quad \therefore m = \frac{1}{\sqrt{2}}$$

$\therefore$ The direction Cosines of one line are $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

If $m=0$, interchanging l, m get the direction Cosines of the other line are $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

$\therefore$ The angle between the two lines

$$= \cos^{-1}\left[\pm\left\{0 \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right\}\right]$$

$$= \cos^{-1}\left[\pm \frac{1}{2}\right]$$

$$= \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$