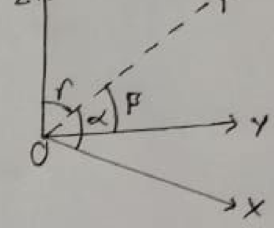
Analytical Geometry of three dimensions

Direction - Cosines :-

Definition :- If α, β, γ be the angles which a given directed line makes with the positive directions of the axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction-cosines of the line.



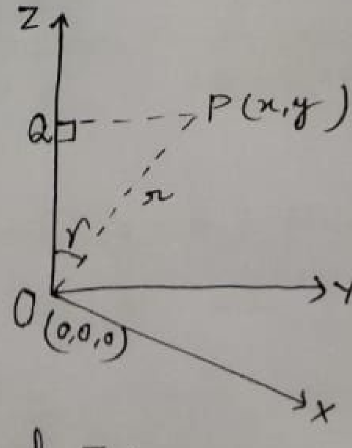
The direction-cosines of a line are usually denoted by l, m, n so that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Theorem :- If l, m, n be the direction cosines of a line, then prove that

$$l^2+m^2+n^2 = 1.$$

Proof :- we know that

$$l = \cos\alpha,\ m = \cos\beta,$$
$$n = \cos\gamma.$$



Let $OP = r$ and P be (x, y, z).

From P draw PQ perpendicular to the axis of Z.

Then

$\angle POQ = \gamma$. Then $OQ = z$.

$\therefore \cos\gamma = \dfrac{OQ}{OP} = \dfrac{z}{r}$

or, $z = r\cos\gamma = rn$.

Similarly

$x = rl$ and $y = rm$.

Now $OP^2 = x^2 + y^2 + z^2$

or, $r^2 = r^2 l^2 + r^2 m^2 + r^2 n^2$

$= r^2(l^2 + m^2 + n^2)$

Hence $l^2 + m^2 + n^2 = 1$.

Note:- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition :- If a, b, c are proportional to the direction - cosines of a line, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$$

Hence if P be the point (a, b, c) then the direction cosines of OP are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

and those of PO are

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}$$

Any three numbers proportional to the direction cosines of a line are called the Direction Ratios of the line.

The direction cosines of a line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.