Alternating Series:-

A series in which the terms are alternately positive or negative is called an alternating series.

Exp. $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \infty$

where $a_n > 0 \ \forall \ n \in N$

Illustrations:-

i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$

ii) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \cdots$

iii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots \cdots$

Leibnitz's Test:-

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \cdots$

converges if (i) $a_{n+1} \leq a_n \ \forall n$, (i.e each term is numerically less than its preceding term and

(ii) $\lim_{n \to \infty} a_n = 0$

<u>Proof</u>:. The given alternating series is

$$a_1 - a_2 + a_3 - a_4 + \cdots\cdots \quad -(1)$$

Suppose $a_1 > a_2 > a_3 > a_4 > \cdots > a_{n+1} \cdots$ — (2)

and $\lim_{n \to \infty} a_n = 0$

In order to prove that the given alternating series converges, its sequence $\langle S_n \rangle$ of partial sums converges, so that the subsequences $\langle S_{2n} \rangle$ and $\langle S_{2n+1} \rangle$ both converge to the same limit.

Now $S_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}$

$S_{2n+2} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} + a_{2n+1} - \cdots - a_{2n+2}$

$\therefore \quad S_{2n+2} - S_{2n} = a_{2n+1} - a_{2n+2} \geq 0$

$\because a_{n+1} \leq a_n \ \forall n$

Thus $\langle S_{2n} \rangle$ is a monotonically increasing sequence.

Again $S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots a_{2n-1} - a_{2n}$

$\because \quad S_n = a_1 - [(a_2 - a_3) + (a_4 - a_5) + \cdots (a_{2n-2} - a_{2n-1}) + a_{2n}]$

Now each term within the bracket is positive, since.

$$a_{n+1} \leq a_n \quad \forall\, n \quad \text{and} \quad a_{2n} > 0$$

$\therefore \quad S_{2n} < a_1 \quad \forall\, n$ and so $\langle S_{2n} \rangle$ is bounded above, therefore $\langle S_{2n} \rangle$ is convergent.

Let $\lim\limits_{n\to\infty} S_{2n} = l$ —— (iii)

Now $\langle S_{2n+1} \rangle$ converges to l. We have

$$S_{2n+1} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} + (-1)\, a_{2n} + a_{2n+1}.$$

or $S_{2n+1} = S_{2n} + a_{2n+1}$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1})$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$$

$$\lim_{n\to\infty} S_{2n+1} = l + 0 \qquad \text{by (iii)}$$

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$$\lim_{n\to\infty} S_{2n+1} = l \quad \text{—— (iv)}$$

From (iii) & (iv) it follows that for any ϵ there exist positive integers m_1 & m_2

$$|S_{2n} - l| < \epsilon, \quad \forall\ n \geq m_1 \qquad \text{---(v)}$$

and $$|S_{2n+1} - l| < \epsilon \quad \forall\ n \geq m_2 \qquad \text{---(vi)}$$

Let $m = \max(m_1, m_2)$ so that

$$m \geq m_1,\ m \geq m_2 \qquad \text{(vii)}$$

From (v), (vi), (vii);

$$|S_n - l| < \epsilon \quad \forall\ n \geq m$$

$\Rightarrow \langle S_n \rangle$ converges to l.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent.

<u>Remark</u>: The alternating series $\sum (-1)^{n} a_n$ is not be convergent if either (i) $a_{n+1} \nleq a_n\ \forall n$ or $\lim_{n\to\infty} a_n \neq 0$

<u>Exp.</u> $1 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$ is not conver

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$