Abstract Algebra

ALGEBRAICALLY CLOSED FIELDS

Definition 1. (Algebraic closure) Let K be an extension of a field F. Then a subfield of all those elements of K which are algebraic over F, is called an algebraic closure of F in K, it is denoted by $\overline{F}$.

Definition 2 (Algebraically closed field). A field F is said to be algebraically closed if every polynomial in $F[x]$ of degree greater than and equal to 1 has a root in F.

For example Q is the algebraic closure of Q in $Q[x]$, but Q is algebraically closed because x^2+1 has no roots in Q.

Theorem 1 A field F is algebraically closed if and only if every irreducible polynomial in $F[x]$ is of degree 1.

Proof:- Let F be algebraically closed and let $f(x) \in F[x]$ be an irreducible polynomial of degree n. Then there exists a finite extension K of F such that $[K:F] = n$.

But F is algebraically closed, so that $K = F$ and hence $[K:F] = 1$, i.e. $n = 1$. Consequently deg. $f(x) = 1$.

Conversely, suppose every irreducible polynomial in $F[x]$ is of degree 1.

Let K be any algebraic extension of F and let $\alpha \in K$, then α is a root of a irreducible polynomial of $F[x]$. But the degree of every irreducible polynomial in $F[x]$ is of degree 1. then $\alpha \in F$. Thus $[K:F] = 1$. Consequently $K = F$ and F is therefore algebraically closed field.

<u>Theorem 2</u>:- A field F is algebraically closed if and only if every polynomial in $F[x]$ of positive degree, factors in $F[x]$ into linear factors.

<u>Proof</u>:- Let F be algebraically closed and let $f(x)$ be a polynomial in $F[x]$. Then $f(x)$ has a root in F.

If this root of $f(x)$ is α, then $(x-\alpha)$ is a factor of $f(x)$, so that $f(x) = (x-\alpha)g(x)$

If $g(x) \in F(x)$ is of positive degree, then it will a root say β in F such that

$$g(x) = (x-\beta)h(x)$$

so we have $f(x) = (x-\alpha)(x-\beta)h(x)$

Continuing the above process, we get a factorization $f(x)$ in $F[x]$ into linear factors.

Conversely, suppose that every polynomial in $F[x]$ of positive degree factors in $F[x]$ into linear factors.

If $(ax-b)$ with $a \neq 0$ is a linear factors of $f(x)$, then b/a is a root of $f(x)$ which belongs to F. Hence F is algebraically closed.