Abstract Algebra

Definition 5 Algebraic extension of a field

Let K be an extension of a field F. Then K is said to be an algebraic extension of F if every element of K is an algebraic over F.

Notes :-

(i) Every finite extension of a field is an algebraic extension.

(ii) An element $\alpha \in K$ is algebraic over F if and only if the degree of $F(\alpha)$ over F is finite.

Theorem :- Transivity of algebraic field extensions

If L is an algebraic extension of K and if K is an algebraic extension of F, then L is an algebraic extension of F.

Proof :- Let α be an arbitrary element of L. Now our aim is to show that α will satisfy a non-zero monic polynomial over F.

Since L is an algebraic over K, and $\alpha \in L$ then α will satisfy a non zero monic polynomial

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

over K i.e. $a_1, a_2, a_3, \ldots, a_n \in K$

But K is an algebraic extension of F, so each element $a_1, a_2, \ldots a_n$ is algebraic over F.

Thus an extension $K = F(a_1, a_2, \cdots, a_n)$ is a finite extension of F. Since α satisfies the polynomial $p(x)$ whose coefficients are in K, therefore, α is algebraic over K, so that $K(\alpha)$ is a finite extension of K.

By the transivity of finite extension fields,

$$[K(\alpha) : F] = [K(\alpha) : F]\,[K : F]$$

since $[K(\alpha) : K]$ and $[K : F]$ are finite, so that $[K(\alpha) : F]$ is finite, i.e. $K(\alpha)$ is a finite extension of F, hence α is algebraic over F.

But α is an arbitrary element of L, hence every element of L is algebraic over F.

Consequently L is an algebraic extension of F.