RINGS

SUBFIELD :-

Definition :- Let K be a non-empty subset of a field F. Then K is a subfield of F if K itself is a field under the same operations as defined on F.

For example :- The field of real numbers is a subfield of the field of complex numbers.

For example :- The field of rational number is a subfield of the field of real numbers.

Theorem :- The necessary and sufficient condition for a non-empty subset K of a field F to be a subfield of F are

(i) $a \in K,\ b \in K \Rightarrow a-b \in K$

(ii) $a \in K,\ b \in K \Rightarrow ab^{-1} \in K$

Proof :- Suppose K is a subfield of F. Then if

$b \in K \Rightarrow -b \in K$

Now, $a \in K,\ -b \in K$.

$\Rightarrow a + (-b) \in K$ ($\because K$ is closed under addition)

$\Rightarrow a-b \in K$

$\Rightarrow$ and if $b \in K \Rightarrow b^{-1} \in K$

(∵ K is a field)

Now $a \in K$, $b^{-1} \in K$

$\Rightarrow ab^{-1} \in K$ (∵ K is closed under multiplication)

Conversely, suppose K is non-empty subset of F and

(i) $a \in K, b \in K \Rightarrow a-b \in K$

(ii) $a \in K, b \in K \Rightarrow ab^{-1} \in K$

Then we have to show that K is a subfield of F. For this, we have from (i)

$a \in K, a \in K$

$\Rightarrow a-a \in K$

$\Rightarrow 0 \in K$

∴ Additive identity exists in K. Also from (i), we have

$0 \in K, a \in K$

$\Rightarrow 0-a \in K$

$\Rightarrow -a \in K$

∴ Additive inverse exists in K. Also, from (ii), we have

$a \in K, a \in K$

$\Rightarrow aa^{-1} \in K$

$\Rightarrow 1 \in K$

$\therefore$ Multiplicative identity exists in K

again from (ii). we have

$1 \in K, \; a \in K$

$\Rightarrow 1 \cdot a^{-1} \in K$

$\Rightarrow a^{-1} \in K$

$\therefore$ Multiplicative inverse exists in K.

Further since we have that addition and multiplication are always distributive over addition. Hence K is a field and hence K is a subfield of F