RINGS

Theorem 3:- A field has no zero divisors.

Proof :- Since field is a Commutative ring with unit element in which every non-zero element possesses its multiplicative inverse. Let $a \in F$ and $a \neq 0$ a (field), then a^{-1} exists in F, is also non-zero. Let us assume

$$a^{-1} = 0$$

$$\Rightarrow aa^{-1} = a0$$

$$\Rightarrow 1 = 0 \quad (\because aa^{-1} = 0,\ a0 = 0)$$

This gives a Contradiction

Thus $a^{-1} \neq 0$ and $aa^{-1} = 1 \neq 0$

Therefore in a field F product of two non-zero element is again a non-zero element. Hence F has no zero divisors.

Notes :- A skew field has no zero divisors.

Theorem 4:– Every finite integral domain is a field.

Proof:– Let D be a finite integral domain. Therefore by the definition of integral domain we have that D is a commutative ring with unit element having no zero divisors. Let $D = \{a_1, a_2, \ldots a_n\}$. In order to show that D is a field we only have to show that D has multiplicative inverse for every non-zero element in D. For this purpose let $a \neq 0$ be any arbitrary element of D and we consider the set

$$D_1 = \{aa_1, aa_2, \ldots, aa_n\}$$

D_1 has n distinct products. For this let us suppose aa_i for $i \neq j$

$$\Rightarrow a(a_i - a_j) = 0 \quad \text{(By left distributive law)}$$

Since D has no zero divisors and $a \neq 0$ then

$$a_i - a_j = 0 \quad \text{for } i \neq j$$

$$\Rightarrow a_i = a_j \quad \text{for } i \neq j$$

$\Rightarrow$ This is a contradiction, because D has n distinct elements $a_1, a_2, \ldots, a_n$. Consequently D_1 has n distinct products. But the elements of D_1 are the elements of D placed in some order. Further D has unit element, that is

$$1 \in D$$

$$\Rightarrow 1 \in D_1 .$$

This implies that there exists an elements b in D such that

$$ab = 1.$$

But D is commutative. Therefore

$$ab = 1 = ba$$

Thus a^{-1} exists in D. Hence D is a field.