# RINGS



## RINGS WITH AND WITHOUT ZERO DIVISORS

Zero divisors :- By the elementary property of a ring we know that if 0 is an additive identity in R (a ring) then $0a = 0 = 0a \ \forall \ a \in R$. But in some Rings it is possible that $ab = 0$ when neither $a = 0$ nor $b = 0$. Then such type of elements a and b are called Zero divisors.

Ring with zero divisors :- A ring R is said to be ring with zero divisor if there exist non-zero elements a,b in R such that $ab = 0$, that is, if $a \neq 0$, $b \neq 0$ but $ab = 0$.

For Example :- The set M of all matrices of order $2 \times 2$ having their elements as integers forms a ring with zero divisors under addition and multiplication of matrices, that is,

if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then $AB = O$ but $A \neq O$ and $B \neq O$

## Ring without zero divisors

A ring R is said to be ring without zero divisor if $ab = 0$, then either $a = 0$ or $b = 0$.

For Example :- The ring of integers is a ring without zero divisors.

## CANCELLATION LAWS IN A RING

Definition :- Let R be a ring and $a, b, c \in R$ : for $a \neq 0$, if $ab = ac$ then $b = c$, and for $a \neq 0$, if $ba = ca$ then $b = c$. The first is known as left Cancellation law while the second is known as right Cancellation law.

Theorem :- A ring R is without zero divisors if and only if the Cancellation laws holds in R.

Proof :- Suppose the ring R is without zero divisors, then we shall have to show that the Cancellation laws hold in R.

Let $a \in R$ and $a \neq 0$ and we have

$$ab = ac$$

$\Rightarrow ab - ac = 0$

$\Rightarrow a(b-c) = 0$ (By left distributive law)

$\Rightarrow b - c = 0$ ($\because$ R is of without zero divisor)

$\Rightarrow b = c$

Similarly $ba = ca$

$\Rightarrow ba - ca = 0$

$\Rightarrow (b-c)a = 0$ (By right distributive law)

$\Rightarrow b - c = 0$

$\Rightarrow b-c = 0$ ($\because$ R is of without zero divisor)

$\Rightarrow b = c$.

Conversely, suppose cancellation laws hold in R, then we have to show that R is without zero divisor. Let us assume R is with zero divisor, that is

$ab = 0$, with $a \neq 0$, $b \neq 0$

$\therefore ab = a0$ ($\because a0 = 0$)

$\Rightarrow b = 0$ (By right cancellation law)

This gives again contradiction. Hence R is without zero divisors.