RINGS

IDEALS :-

Definition (left ideal):- An additive subgroup S of a ring R is said to be a left ideal of R if

$$a \in S,\ r \in R \Rightarrow ra \in S$$

Definition (Right ideal) :- An additive subgroup S of a ring R is said to be a right ideal of R if

$$a \in S,\ r \in R \Rightarrow ar \in S$$

Definition (Ideal) :- A non empty subset S of a ring R is said to be an ideal of R if it is both left and right ideal. That is, an additive subgroup S of R is an ideal of R if

$$a \in S,\ r \in R \Rightarrow ra \in S,\ ar \in S.$$

In particular, the subset $\{0\}$ consisting of 0 alone and the ring R itself are ideals in R. These two ideals $\{0\}$ and R are called the improper ideals of R; all other ideals of R are called proper.

Notes:-

① A two-sided ideal is called simply an ideal.

Theorems on Ideals :-

Theorem :- The necessary and sufficient Conditions for a non-empty subset S of a ring R to be an ideal of R are

(i) $a \in S,\ b \in S \Rightarrow a-b \in S$

(ii) $a \in S,\ r \in R \Rightarrow ra \in S$ and $ar \in S$

Proof :- The Conditions are necessary. Let S be an ideal of R, then by definition of an ideal, S is additive subgroup of R and $ar \in S$ and $ra \in S$ for all $a \in S$ and $r \in S$.

But the necessary and sufficient Condition for a non-empty subset S to be a subgroup of an additive group R is

$$a \in S,\ b \in S \Rightarrow a-b \in S$$

Hence, $a \in S,\ b \in S \Rightarrow a-b$ and

$$a \in S,\ r \in S \Rightarrow ra \in S \text{ and } ar \in S$$

The Conditions are sufficient :- Let S be any non-empty subset of a ring R such that

(i) $a \in S,\ b \in S \Rightarrow a-b \in S$

(ii) $a \in S,\ r \in S \Rightarrow ra \in S$ and $ar \in S$

If a and b be any two elements of S, then from the Condition (i), we have

$$a \in S,\ b \in S \Rightarrow a-b \in S$$

And from Condition (ii), we have

$$a \in S,\ b \in R \Rightarrow ab \in S$$

Thus, $a \in S,\ b \in S \Rightarrow a-b \in S$ and $ab \in S$ implying that S is an additive subgroup of R because the condition $a \in S, b \in S \Rightarrow a-b \in S$ and $ab \in S$ is a necessary and sufficient condition for a non empty subset S to be a subgroup. Therefore, S is an additive subgroup of R and $ra \in S$ and $ar \in S$ for all $a \in S$ and $r \in R$. Hence, S is an ideal of R.

Theorem :- The intersection of any two ideals of a ring R is again an ideal of R.

Proof :- Let S_1 and S_2 be any two ideals of a ring R, then we have to show that $S_1 \cap S_2$ is also an ideal of R.

since S_1 and S_2 are ideals of R so that S_1 and S_2 both are additive subgroups of R. But intersection of two subgroups is also a subgroup of R. Thus $S_1 \cap S_2$ is also an additive subgroup of R.

Now,

$$a \in S_1 \cap S_2 \Rightarrow a \in S_1 \text{ and } a \in S_2$$

$$\therefore\ a \in S_1,\ r \in R \Rightarrow ra \in S_1 \text{ and } ar \in S_1$$
($\because$ S_1 being an ideal)

and $a \in S_2,\ r \in R \Rightarrow ra \in S_2$ and $ar \in S_2$
($\because$ S_2 being an ideal)

Thus, $a \in S_1 \cap S_2,\ r \in R \Rightarrow ra \in S_1 \cap S_2$ and $ar \in S_1 \cap S_2$

Hence, $S_1 \cap S_2$ is an additive subgroup of R and $ra \in S_1 \cap S_2$ and $ar \in S_1 \cap S_2$ for all $a \in S_1 \cap S_2$ and $r \in R$. Consequently $S_1 \cap S_2$ is an ideal of R.