Rings

Theorem :- The characteristic of an integral domain is zero or $n > 0$ according as the order of any non-zero element of the integral domain regarded as a member of additive group of the integral domain either zero or n.

Proof :- Let D be an integral domain and let a be any non-zero element of D.

If the order of a is zero, then obviously the characteristic of D is zero. We suppose that the order of a is $n > 0$. then we have

$$na = 0 \quad \text{—— (1)}$$

Furthermore, suppose b is any other non-zero element of D.

Multiplying both sides of (1) by b, we have

$$(na)b = 0 \cdot b$$

$$\Rightarrow [a + a + a + \cdots + a \text{ (n times)}]b = 0$$

$$(\because 0 \cdot b = 0)$$

$$\Rightarrow ab + ab + ab + \cdots + ab \text{ (n times)} = 0$$

(By right distributive law)

$$\Rightarrow a[b + b + b + \cdots + b \text{ (n times)}] = 0$$

(By left distributive law)

$$\Rightarrow a(nb) = 0$$
$$\Rightarrow nb = 0$$

But $o(a) = n \Rightarrow n$ is the least positive integer such that $nx = 0 \ \forall x = D$.

Hence, the characteristic of D is n.

Theorem :- The characteristic of an integral domain is either zero or a prime number.

Proof :- Let D be an integral domain and let a be any non-zero element of D. If the order of a is zero, then the characteristic of D is zero.

Let us suppose now that the order of a is p. then the characteristic of D is p.

Now we shall show that p is prime. Let if possible p is not prime. then

$p = p_1 p_2$ with $p_1 < p$, $p_2 < p$ and $p_1 \neq 1$, $p_2 \neq 1$.

Since $a \neq 0$ so that $a^2 \neq 0$ and the characteristic of D is p. then

$$p(a^2) = 0$$
$$\Rightarrow p_1 p_2 (a^2) = 0 \qquad (\because p = p_1 p_2)$$

$$\Rightarrow (p_1 a)(p_2 a) = 0$$

$\Rightarrow$ either $p_1 a = 0$ or $p_2 a = 0$

[$\because$ D has no zero divisor]

$\Rightarrow$ the characteristic of D is p, which is the least positive integer therefore neither $p_1 a = 0$ nor $p_2 a = 0$. this implies that p is a prime number.

Hence the theorem.

Theorem :-