RINGS

INTEGRAL DOMAIN, FIELD AND SKEW FIELDS (division ring):-

Integral domain :- A Commutative ring R with unit element having no zero divisors is called an integral domain.

For Example :- The ring of integers $(Z, '+', '\cdot')$ is an integral domain.

For Example :- Let $S = \{a + b\sqrt{2} : a, b$ are real numbers$\}$ then $(S, '+', '\cdot')$ is an integral domain.

Field :- A Commutative ring R with unit element having at least two elements is called a field if every non-zero elements of R possess their multiplicative inverse.

For Example :- The ring of rational numbers $(Q, '+', '\cdot')$ is a field.

For Example :- Let $S = (\{0, 1, 2, 3, 4\}, +_5, \times_5)$, then S is a finite field.

RINGS

Integral domain, Field and skew-fields :-

Integral domain :- A Commutative ring R with unit element having no zero divisors is called an integral domain.

Example :- The ring of integers $(Z, +, \cdot)$ is an integral domain.

Some Important Theorems

Theorem 1 :- Every field is an integral domain.

Proof :- Let F be a field. we have to show that F is an integral domain. Since F is a field so it is Commutative ring with unit element. Therefore in order to show F to be an integral domain, we only have to show that F has no zero divisors.

For this

Let $a \in F$ and $a \neq 0$ then a^{-1} exists in F. We have

$$ab = 0$$

$$\Rightarrow a^{-1}(ab) = a^{-1}0$$

$$\Rightarrow (a^{-1}a)b = a^{-1}0 \quad \text{(By associative law)}$$

$$\Rightarrow 1b = a^{-1}0 \quad (\because a^{-1}a = 1)$$

$$\Rightarrow 1b = 0 \quad \text{(By elementary property of ring)}$$

$\Rightarrow b = 0$

$(\because 1 \cdot a = a = a \cdot 1)$

Similarly, let $b \in F$ and $b \neq 0$ and $ab = 0$

$\Rightarrow (ab)b^{-1} = 0b^{-1}$

$\Rightarrow a(bb^{-1}) = 0$ By associative law and $0b^{-1} = 0 = b^{-1}0$

$\Rightarrow a1 = 0$ $(\because bb^{-1} = 1)$

$\Rightarrow a = 0$ $(\because a \cdot 1 = a = 1 \cdot a)$

Thus we obtained that in F, $ab = 0$, then either $a = 0$ or $b = 0$. Hence F is an integral domain.

Theorem 2 :- Disprove that every integral domain is a field.

Proof :- To disprove that every integral domain is a field we shall give an example. Since we know that the ring of integers is an integral domain but it is not field, because if $a \in I$ and $a \neq 0$, then $a^{-1} \notin I$.