* Adaptive Quadrature Methods:-

A numerical integration procedure which adopts automatically a suitable step-size (interval h) to solve an integration problem numerically is called Adaptive Quadrature Method. Here we describe an adaptive quadrature method based on Simpson's (1/3) rule.

Suppose that

$$I = \int_a^b y\,dx \quad \text{—— (1)}$$

to within an accuracy $\epsilon > 0$. Using Simpson's (1/3) rule with $h = (b-a)/2$, we obtain

$$I = \int_a^b y\,dx = \frac{h}{3}\left[y(a) + 4y\left(\frac{a+b}{2}\right) + y(b)\right] - \frac{h^5}{90} y^{iv}(\xi_1)$$

$$a < \xi_1 < b$$

$$= I(a,b) - \frac{(b-a)h^4}{180} y^{iv}(\xi_1) \quad \text{—— (2)}$$

where

$$I(a,b) = \frac{h}{3}\left[y(a) + 4y\left(\frac{a+b}{2}\right) + y(b)\right] \text{—— (3)}$$

Now we subdivided the interval and set $h = \frac{b-a}{4}$ then Simpson's (1/3) rule gives

$$I = \int_a^b y\,dx = \frac{h}{6}\left[y(a) + 4y\frac{3a+b}{4} + 2y\frac{a+b}{2} + 4y\frac{a+3b}{4} + y(b)\right] - \frac{h^4(b-a)}{180\times 16} y^{iv}(\xi_2)$$

$$= \frac{h}{6}\left[y(a) + 4y\left(\frac{3a+b}{4}\right) + y\frac{(a+b)}{2}\right]$$

$$+ \frac{h}{6}\left[y\frac{(a+b)}{2} + 4y\frac{a+3b}{4} + y(b)\right]$$

$$- \frac{(b-a)}{180\times 16} h^4 y^{iv}(\xi_2)$$

$$\int_a^b y\,dx = I = I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right) - \frac{(b-a)h^4}{180\times16} y^{iv}(\xi_2)$$ —— (4)

where

$$I\left(a, \frac{a+b}{2}\right) = \frac{h}{6}\left[y(a) + 4y\frac{3a+b}{4} + y\frac{(a+b)}{2}\right]$$

and $$I\left(\frac{a+b}{2}, b\right) = \frac{h}{6}\left[y\frac{(a+b)}{2} + 4y\frac{(a+3b)}{4} + y(b)\right]$$ —— (5)

Assuming

$$y^{iv}(\xi_1) = y^{iv}(\xi_2)$$

from Eqn (3) and (4) give on simplification

$$\frac{1}{15}\left[I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right)\right] = \frac{(b-a)}{180\times16} h^4 y^{iv}(\xi_2)$$ —— (6)

on substituting Eqn (6) in Eqn (4) we get

$$\left|\int_a^b y\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right)\right|$$

$$= \frac{1}{15}\left|\left[I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right)\right]\right|$$

if we suppose that

$$\frac{1}{15}\left|\left[I(a,b) - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right)\right]\right| < \epsilon$$

then

$$\left|\int_a^b y\,dx - I\left(a, \frac{a+b}{2}\right) - I\left(\frac{a+b}{2}, b\right)\right| < \epsilon$$

Hence

$$\int_a^b y\,dx \simeq I\left(a, \frac{a+b}{2}\right) + I\left(\frac{a+b}{2}, b\right)$$

to within an accuracy of $\epsilon > 0$

If the inequality is not satisfied then the procedure is applied to each intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ with $\epsilon/2$.

Exp Test the error in the evaluation of the integ-ral $I = \int_0^{\pi/2} \cos x \, dx$ using $I(a,b) = \frac{h}{3}\left[y(0) + 4y\frac{a+b}{2} + \ldots\right.$

Sol. Let $h = \pi/4$ then $a = 0$, $b = \pi/2$

$$I(0, \pi/2) = \frac{h}{3}\left[\cos 0 + 4\cos\frac{\pi}{4} + \cos\pi/2\right]$$

$$I(0, \pi/2) = \frac{\pi}{12}\left[1 + 4 \times \frac{1}{\sqrt{2}} + 0\right] = 1.00228$$

$$I(0, \pi/4) = \frac{\pi}{24}\left[\cos 0 + 4\cos\frac{\pi}{8} + \cos\pi/4\right]$$

$$\rightarrow I(0, \pi/4) = \frac{\pi}{24}\left(1 + 4\cos\pi/8 + \frac{1}{\sqrt{2}}\right]$$

$$I(\pi/4, \pi/2) = \frac{\pi}{24}\left[\cos\pi/4 + 4\cos\frac{3\pi}{8} + \cos\pi/2\right]$$

$$\rightarrow I(\pi/4, \pi/2) = \frac{\pi}{24}\left[\frac{1}{\sqrt{2}} + 4\cos\frac{3\pi}{8} + 0\right]$$

Then $I(0, \frac{\pi}{4}) + I(\frac{\pi}{4}, \pi/2) = \frac{\pi}{24}\left[1 + \frac{2}{\sqrt{2}} + 4\cos\frac{\pi}{8} + 4\ldots\right.$

$$= \frac{\pi}{24}\left[1 + \sqrt{2} + 4\left(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8}\right)\right]$$

$$= 1.00013$$

and $\frac{1}{15}\left[I(0, \pi/2) - I(0, \pi/4) - I(\frac{\pi}{4}, \pi/2)\right]$

$$= \frac{1}{15}\left[0.00215\right] = 0.00014$$

Hence Actual error

$$= \left| \int_0^{\pi/2} \cos x \, dx - I(0, \pi/4) - I(\frac{\pi}{4}, \pi/2) \right| = 0.00013$$

which is less than that obtain 0.00014.