Abstract Algebra

Prime ideals

Definition :- If R be a Commutative ring then, an ideal $S \neq R$ is said to be prime ideal if $ab \in S$ implies that either $a \in S$ or $b \in S$ for all $a, b \in R$

For example In a ring ideal $P = \{pn : n \in Z$, where p is prime$\}$ is a prime ideal of Z.

Theorem Let R be a Commutative ring with unity and let $S \neq R$ be an ideal of R. Then R/S is an integral domain if and only if S is a prime ideal of R.

Proof :- Since R is a commutative ring with unity so that R/S is an integral domain if and only if S is a prime ideal of R. ~~P~~ with Commutative ring with unity. Therefore, in order to show that R/S is an integral domain, we shall show that R/S has no zero divisors.

Let us suppose that S is a prime ideal of R and let $a, b \in R$, then $S+a$ and $S+b$ be any arbitrary elements of R/S.

If $(S+a)(S+b) = S$ ($\because$ zero element of R/S is S)

$\Rightarrow S+(ab) = S$

$\Rightarrow ab \in S$

$\Rightarrow$ either $a \in S$ or $b \in S$

$\Rightarrow$ either $S+a = S$ or $S+b = S$

$\Rightarrow R/S$ has no zero divisors

$\Rightarrow R/S$ is an integral domain.

Conversely, R/S is an integral domain so that if has no zero divisors.

Therefore, if $S+a, S+b \in R/S \ \forall a, b \in R$ then

$$(S+a)(S+b) = S$$

$\Rightarrow$ either $S+a = S$ or $S+b = S$

$\Rightarrow$ either $a \in S$ or $b \in S$

But $(S+a)(S+b) = S \Rightarrow S+ab = S$

$\Rightarrow ab \in S$

Thus, if $ab \in S$, then either $a \in S$ or $b \in S$. Hence, S is a prime ideal of R

Maximal Ideals

Definition A proper ideal S of a ring R is said to be maximal ideal of R if it is not strictly contained by any other proper ideal of R.

OR

Definition A proper ideal S of ring R is said to be maximal ideal of S if T is any other proper ideal of R such that either $S = T$ or $T = R$.

Definition A proper ideal S of a ring R is said to be maximal ideal of R if there exists no proper ideal between S and R.

For example :- If R is a ring of even integers, then the ideal $(4) = \{4m : m \in Z\}$ is maximal ideal of R because $2 \notin (4)$ and $4 \neq R$.

Notes :-

(i) If S be an ideal of a ring Z of all integers. Then S is maximal if and only if it is generated by some prime integer.

(ii) An ideal S of a commutative ring R with unity is maximal if and only if the residue class ring R/S is a field.

(iii) An ideal generated by a single element is called principal ideal and the ring whose every ideal is a principal ideal is called principal ideal ring.

(iv) A ring R is a field if and only if its zero ideal is a maximal ideal.

(v) Let R be a commutative ring with unity. Then every maximal ideal of R is a prime ideal.